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Niezależne od wymiaru szacowania transformat Riesza związanych z oscylatorem harmonicznym

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DIMENSION-FREE ESTIMATES FOR RIESZ TRANSFORMS RELATED TO THE HARMONIC OSCILLATOR

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Abstract. We study L^p bounds for two kinds of Riesz transforms on $\mathbb{R}^d$ related to the harmonic oscillator. We pursue an explicit estimate of their L^p norms that is independent of the dimension d and linear in $\max(p, p/(p-1))$.

1. Introduction

The aim of this paper is to prove a dimension-free estimate for the L^p norm of vectors of a specific kind of generalized Riesz transforms. Recall that the classical Riesz transforms on $\mathbb{R}^d$ are the operators

$$R_i f(x) = \partial_{x_i} (-\Delta)^{-1/2} f(x), \quad i = 1, \dots, d.$$

A well-known result concerning Riesz transforms, proved by Stein in [14], is the L^p boundedness of the vector of the Riesz transforms

$$\mathbf{R}f = (R_1 f, \dots, R_d f)$$

with a norm estimate independent of d. Since then, the question about dimension-free estimates for the Riesz transforms has been asked in various contexts. For example Carbonaro and Dragičević proved in [1] a dimension-free estimate with an explicit constant for the shifted Riesz transform on a complete Riemannian manifold. Another path of generalizing the result of Stein is to consider operators of the form

$$R_i = \delta_i L^{-1/2}, \tag{1.1}$$

where δ_i is an operator on $L^2(\mathbb{R}^d)$ and

$$L = \sum_{i=1}^{d} L_i = \sum_{i=1}^{d} (\delta_i^* \delta_i + a_i), \quad a_i \geqslant 0.$$

Such Riesz transforms were studied systematically by Nowak and Stempak in [13]. We will focus on the Riesz transforms of the form as in (1.1) where L is the harmonic

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oscillator ($L = -\Delta + |x|^2$), i.e.

$$\delta_i = \partial_{x_i} + x_i, \quad \delta_i^* = -\partial_{x_i} + x_i, \quad a_i = 1. \tag{1.2}$$

From this point δ_i and δ_i^* are defined as above.

This so-called *Hermite-Riesz transform* was introduced by Thangavelu in [15], who proved its L^p boundedness. Then a dimension-free estimate of its norm was proved in [7] and [8], which later was sharpened by Dragičević and Volberg in [5] to an estimate linear in $\max(p, p/(p-1))$.

In the first part we will give a result analogous to Theorem 10 from [16], however concerning a slightly altered operator, namely

$$R_i' = \delta_i^* L'^{-1/2}$$

with

$$L_i' = \delta_i \delta_i^* + 1, \quad L' = \sum_{i=1}^{d} L_i'.$$

It arises as a result of swapping δ_i and δ_i^* in the definition of $R_i = \delta_i L^{-1/2}$. As explained in Section 3, the results from [16] do not apply to this operator. The key step in the proof is, as in [16], the method of Bellman function but we use its more subtle properties to achieve the goal.

In the second part we consider the vector of the Riesz transforms

$$\tilde{\mathbf{R}}f = \left(\tilde{R}_1 f, \dots, \tilde{R}_d f\right),$$

where

$$\tilde{R}_i = \delta_i^* L^{-1/2}.$$

Its boundedness was proved in [5] (where $\tilde{R}_i$ was denoted by R_i^*), [7] and [8] with an implicit constant independent of the dimension. Our goal is to give an explicit constant. Due to reasons explained in Section 4 we will focus on proving the boundedness of the operator S defined as

$$Sf(x) = |x| L^{-1/2} f(x).$$

We obtain it by an explicit estimate of the kernel of S. As a corollary we get a dimension-free estimate of the norm of the vector of the operators

$$R_i^* = \delta_i^* (L+2)^{-1/2}$$

with each R_i^* being the adjoint of $R_i = \delta_i L^{-1/2}$ studied in [5] and [16].

2. Preliminaries

In order to define the operators L', L, R_i' and $\tilde{R}_i$ on $L^2(\mathbb{R}^d)$ (later abbreviated as L^2) we introduce the Hermite polynomials and the Hermite functions. The Hermite polynomials are given by

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}, \ x \in \mathbb{R}$$

or, equivalently, by

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \quad n \geqslant 2, \ x \in \mathbb{R},$$
$$H_0(x) = 1, \ H_1(x) = 2x.$$

The Hermite functions are

$$h_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x), \ x \in \mathbb{R}.$$

It is well known that the Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$ and that their linear span is dense in $L^p(\mathbb{R})$ for every $1 \leqslant p < \infty$.

For $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ with $\mathbb{N} = \{0, 1, 2 \dots\}$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define

$$h_n(x) = h_{n_1}(x_1) \cdots h_{n_d}(x_d).$$

We can see that $\{h_n\}_{n \in \mathbb{N}^d}$ is an orthonormal basis of L^2. Throughout the paper we will use $\mathcal{D} = \operatorname{lin}\{h_n : n \in \mathbb{N}^d\} = \operatorname{lin}\{\delta_i^* h_n : n \in \mathbb{N}^d\}$.

Let L' be the operator given on $C_c^\infty(\mathbb{R}^d)$ by

$$L' = \sum_{i=1}^d L_i', \quad L_i' = \delta_i \delta_i^* + 1, \quad \delta_i = \partial_{x_i} + x_i.$$

In a similar way we define on $C_c^\infty(\mathbb{R}^d)$

$$L = \sum_{i=1}^d L_i, \quad L_i = \delta_i^* \delta_i + 1.$$

Since $\delta_i \delta_i^* = \delta_i^* \delta_i + 2$, we can also write

$$L' = L + 2d. \tag{2.1}$$

Note that the formal adjoint of δ_i with respect to the inner product on L^2 is $\delta_i^* = -\partial_{x_i} + x_i$. We recall well-known relations concerning the Hermite functions.

Lemma 1. *For $n \in \mathbb{N}^d$ and $i = 1, \dots, d$ we have*

1. $\delta_i h_n(x) = \begin{cases} \sqrt{2n_i} h_{n-e_i}(x) & \text{if } n_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$

2. $\delta_i^* h_n(x) = \sqrt{2(n_i+1)}h_{n+e_i}(x)$,
3. $L_i' h_n(x) = (2n_i+3)h_n(x)$,
4. $L_i h_n(x) = (2n_i+1)h_n(x)$.

Hence, the multivariate Hermite functions $\{h_n\}_{n\in\mathbb{N}^d}$ are eigenvectors of L' and L corresponding to positive eigenvalues $\{\lambda_n'\}_{n\in\mathbb{N}^d}$ and $\{\lambda_n\}_{n\in\mathbb{N}^d}$ respectively, where $\lambda_n' = 2|n|_1 + 3d$, $\lambda_n = 2|n|_1 + d$ with $|n|_1 = n_1 + \cdots + n_d$ for $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. It is well known that L (and L') are essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ with the self-adjoint extensions given by

$$L'f = \sum_{n\in\mathbb{N}^d} \lambda_n' \langle f, h_n\rangle h_n, \quad Lf = \sum_{n\in\mathbb{N}^d} \lambda_n \langle f, h_n\rangle h_n,$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product, acting on the domains

$$\mathrm{Dom}(L') = \{f \in L^2 : \sum_{n\in\mathbb{N}^d} \lambda_n'^2 |\langle f, h_n\rangle|^2 < \infty\},$$

$$\mathrm{Dom}(L) = \{f \in L^2 : \sum_{n\in\mathbb{N}^d} \lambda_n^2 |\langle f, h_n\rangle|^2 < \infty\}.$$

Then $R_i' = \delta_i^* L'^{-1/2}$ can be defined rigorously as

$$R_i' f = \sum_{n\in\mathbb{N}^d} \lambda_n'^{-1/2} \langle f, h_n\rangle \delta_i^* h_n$$

and $\tilde{R}_i = \delta_i^* L^{-1/2}$ as

$$\tilde{R}_i f = \sum_{n\in\mathbb{N}^d} \lambda_n^{-1/2} \langle f, h_n\rangle \delta_i^* h_n.$$

It is clear that R_i' and $\tilde{R}_i$ are bounded on L^2.

In what follows we will often identify a densely defined bounded operator on a Banach space with its unique bounded extension to the whole space. As for the notation, we will abbreviate

$$L^p = L^p(\mathbb{R}^d), \quad \|\cdot\|_p = \|\cdot\|_{L^p} \quad \text{and} \quad \|\cdot\|_{p\to p} = \|\cdot\|_{L^p\to L^p}$$

and for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we will use $|x| = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$. For $1 < p < \infty$ we denote $p^* = \max\left(p, \frac{p}{p-1}\right)$.

3. Riesz transforms of the first kind

Let $\mathbf{R}'f = (R_1'f, \ldots, R_d'f)$. The main result of this section gives an explicit estimate for the L^p norm of $\mathbf{R}'$.

Theorem 2. *For $f \in L^p$ we have*

$$\|\mathbf{R}'f\|_p := \left(\int_{\mathbb{R}^d} |\mathbf{R}'f(x)|^p \, dx \right)^{1/p} \leqslant 48(p^* - 1)\|f\|_p.$$

In order to prove Theorem 2, we will need some auxiliary objects. One can see that $L_i' = -\partial_{x_i}^2 + x_i^2 + 2$, so we can write

$$-\Delta = -\sum_{i=1}^{d} \partial_{x_i}^2 = L' - r, \qquad \text{where } r(x) = |x|^2 + 2d.$$

We will also need the operators M_i defined on $C_c^\infty(\mathbb{R}^d)$ as

$$M_i = \sum_{j \neq i} \delta_j \delta_j^* + \delta_i^* \delta_i = L' + [\delta_i^*, \delta_i] = L' - 2,$$

where

$$[\delta_i^*, \delta_i] = \delta_i^* \delta_i - \delta_i \delta_i^*.$$

Note that in our case $[\delta_i^*, \delta_i] = -2 < 0$. This means that the crucial assumption from [16] does not hold and the theory does not apply.

Non-zero elements of $\{c_n^i \delta_i^* h_n\}_{n \in \mathbb{N}^d}$ (where c_n^i are the normalizing constants) form an orthonormal system of eigenvectors of M_i with eigenvalues $\{\lambda_n'\}_{n \in \mathbb{N}^d}$. Thus, we can define the self-adjoint extensions of M_i by

$$M_i f = \sum_{n \in \mathbb{N}^d} \lambda_n' \left\langle f, c_n^i \delta_i^* h_n \right\rangle c_n^i \delta_i^* h_n$$

on the domain

$$\operatorname{Dom}(M_i) = \{f \in L^2 : \sum_{n \in \mathbb{N}^d} \lambda_n'^2 \left| \left\langle f, c_n^i \delta_i^* h_n \right\rangle \right|^2 < \infty\}.$$

Having these operators, we can introduce the semigroups

$$P_t = e^{-tL'^{1/2}} \qquad \text{and} \qquad Q_t^i = e^{-tM_i^{1/2}}$$

rigorously defined as

$$P_t f = \sum_{n \in \mathbb{N}^d} e^{-t\lambda_n'^{1/2}} \langle f, h_n \rangle h_n, \quad Q_t^i f = \sum_{n \in \mathbb{N}^d} e^{-t\lambda_n'^{1/2}} \left\langle f, c_n^i \delta_i^* h_n \right\rangle c_n^i \delta_i^* h_n.$$

Lemma 3. *Let $i = 1, \ldots, d$. If $f, g \in \mathcal{D}$, then*

$$\langle R_i' f, g \rangle = -4 \int_0^\infty \left\langle \delta_i^* P_t f, \partial_t Q_t^i g \right\rangle t \, dt.$$

Proof. The proof is analogous to the proof of Proposition 3 in [16] but we give it for the sake of completeness. By linearity it is sufficient to prove the lemma for $f = h_n$ and $g = \delta_i^* h_k$ for some $n, k \in \mathbb{N}^d$. We proceed as follows:

$$\begin{aligned}
-4 \int_0^\infty \left\langle \delta_i^* P_t h_n, \partial_t Q_t^i \delta_i^* h_k \right\rangle \, t \, dt &= -4 \int_0^\infty \left\langle e^{-t\lambda_n'^{1/2}} \delta_i^* h_n, -\lambda_k'^{1/2} e^{-t\lambda_k'^{1/2}} \delta_i^* h_k \right\rangle t \, dt \\
&= 4\lambda_k'^{1/2} \left\langle \delta_i^* h_n, \delta_i^* h_k \right\rangle \int_0^\infty e^{-t(\lambda_n'^{1/2} + \lambda_k'^{1/2})} \, t \, dt \\
&= \frac{4\lambda_k'^{1/2}}{\left(\lambda_n'^{1/2} + \lambda_k'^{1/2}\right)^2} \left\langle \delta_i^* h_n, \delta_i^* h_k \right\rangle .
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\left\langle \delta_i^* L'^{-1/2} h_n, \delta_i^* h_k \right\rangle + 4 \int_0^\infty \left\langle \delta_i^* P_t h_n, \partial_t Q_t^i \delta_i^* h_k \right\rangle t \, dt \\
&= \lambda_n'^{-1/2} \left\langle \delta_i^* h_n, \delta_i^* h_k \right\rangle + \frac{4\lambda_k'^{1/2}}{\left(\lambda_n'^{1/2} + \lambda_k'^{1/2}\right)^2} \left\langle \delta_i^* h_n, \delta_i^* h_k \right\rangle \\
&= \left(\lambda_n'^{-1/2} - \frac{4\lambda_k'^{1/2}}{\left(\lambda_n'^{1/2} + \lambda_k'^{1/2}\right)^2} \right) \left\langle \delta_i^* h_n, \delta_i^* h_k \right\rangle .
\end{aligned}$$

If $\lambda_n' = \lambda_k'$, then the expression in parentheses is 0, otherwise $\delta_i^* h_n$ and $\delta_i^* h_k$ — eigenvectors of M_i — are orthogonal. □

We will also need a bilinear embedding theorem. First, for $f = (f_1, \ldots, f_N) : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^N$ we set

$$\begin{aligned}
|f(x,t)|_*^2 = r(x) &\left| (f_1(x,t), \ldots, f_N(x,t)) \right|^2 \\
&+ \left| (\partial_t f_1(x,t), \ldots, \partial_t f_N(x,t)) \right|^2 \\
&+ \sum_{i=1}^d \left| (\partial_{x_i} f_1(x,t), \ldots, \partial_{x_i} f_N(x,t)) \right|^2.
\end{aligned}$$

We also define two auxiliary functions F and G. For $f \in \mathcal{D}$ and $g = (g_1, \ldots, g_d)$ with $g_i \in \mathcal{D}$ let

$$F(x,t) = P_t f(x) \quad \text{and} \quad G(x,t) = Q_t g(x) = \left(Q_t^1 g_1(x), \ldots, Q_t^d g_d(x) \right).$$

Theorem 4. *Take $d \geqslant 2$. Then we have*

$$\int_0^\infty \int_{\mathbb{R}^d} |F(x,t)|_* |G(x,t)|_* \, dx \, t \, dt \leqslant 6(p^* - 1) \|f\|_p \|g\|_q.$$

3.1. **The Bellman function.** In order to prove Theorem 4, let us introduce the Bellman function. Take $p \geqslant 2$ and let q be its conjugate exponent. Define $\beta : [0,\infty)^2 \to [0,\infty)$ by

$$\beta(s,t) = s^p + t^q + \gamma \begin{cases} s^2 t^{2-q} & \text{if } s^p \leqslant t^q \\ \frac{2}{p} s^p + \left(\frac{2}{q} - 1\right) t^q & \text{if } s^p \geqslant t^q \end{cases}, \quad \gamma = \frac{q(q-1)}{8}.$$

The Nazarov–Treil Bellman function is then the function

$$B(\zeta, \eta) = \tfrac{1}{2} \beta \left(|\zeta|, |\eta|\right), \quad \zeta \in \mathbb{R}^{m_1}, \eta \in \mathbb{R}^{m_2}.$$

It was introduced by Nazarov and Treil in [11] and then simplified and used by Carbonaro and Dragičević in [1, 2] and by Dragičević and Volberg in [3, 4, 5]. Note that B is differentiable but not smooth, so we convolve it with a mollifier ψ_κ to get $B_\kappa = B * \psi_\kappa$, where

$$\psi_\kappa(x) = \frac{1}{\kappa^{m_1+m_2}} \psi\left(\frac{x}{\kappa}\right) \quad \text{and} \quad \psi(x) = c_{m_1,m_2} e^{-\frac{1}{1-|x|^2}} \chi_{B(0,1)}(x), \quad x \in \mathbb{R}^{m_1+m_2}$$

and c_{m_1,m_2} is the normalizing constant. The functions B and ψ_κ are biradial and so is B_κ, hence there exists $\beta_\kappa : [0,\infty)^2 \to [0,\infty)$ such that

$$B_\kappa(\zeta,\eta) = \tfrac{1}{2} \beta_\kappa \left(|\zeta|, |\eta|\right).$$

We invoke some properties of β_κ and B_κ that were proved in [5] and [9].

Theorem 5. *Let $\kappa \in (0,1)$ and $s,t > 0$. Then we have*

1. $0 \leqslant \beta_\kappa(s,t) \leqslant (1+\gamma)\left((s+\kappa)^p + (t+\kappa)^q\right)$,
2. $0 \leqslant \partial_s \beta_\kappa(s,t) \leqslant C_p \max\left((s+\kappa)^{p-1}, t+\kappa\right)$,
   $0 \leqslant \partial_t \beta_\kappa(s,t) \leqslant C_p (t+\kappa)^{q-1}$.

The function B_κ is smooth and for every $z = (x,y) \in \mathbb{R}^{m_1+m_2}$ there exists $\tau_\kappa > 0$ such that for $\omega = (\omega_1, \omega_2) \in \mathbb{R}^{m_1+m_2}$ we have

3. $\langle \mathrm{Hess}(B_\kappa)(z)\omega, \omega \rangle \geqslant \frac{\gamma}{2}\left(\tau_\kappa |\omega_1|^2 + \tau_\kappa^{-1} |\omega_2|^2\right)$.

There is a continuous function $E_\kappa : \mathbb{R}^{m_1+m_2} \to \mathbb{R}$ such that

4. $\langle \nabla B_\kappa(z), z \rangle \geqslant \frac{\gamma}{2}\left(\tau_\kappa |x|^2 + \tau_\kappa^{-1} |y|^2\right) - \kappa E_\kappa(z) + B_\kappa(z)$,
5. $|E_\kappa(z)| \leqslant C_{m_1,m_2,p}\left(|x|^{p-1} + |y| + |y|^{q-1} + \kappa^{q-1}\right)$.

3.2. **Proof of Theorem 4.** Having defined the Bellman function, we proceed to the proof. First we should emphasize that the presence of the term $B_\kappa(z)$ in 4. is the key ingredient for the Bellman method to work despite the fact that $[\delta_i^*, \delta_i] < 0$. Because of that, the proof of Lemma 6 is more involved than in [16].

Let

$$u(x,t) = (P_t f(x), Q_t g(x)) = \left(P_t f(x), Q_t^1 g_1(x), \dots, Q_t^d g_d(x)\right)$$

for $x \in \mathbb{R}^d$ and $t > 0$ and fix $p \geqslant 2$. We will use the Bellman function B_κ and $b_\kappa = B_\kappa \circ u$ with $m_1 = 1$ and $m_2 = d$. Our aim is to estimate the integral

$$I(n,\varepsilon) = \int_0^\infty \int_{X_n} \left(\partial_t^2 + \Delta\right)(b_{\kappa(n)})(x,t)\, dx\, te^{-\varepsilon t}\, dt,$$

where $\kappa(n)$ is a number depending on n and $X_n = [-n,n]^d$ so that $\{X_n\}_{n\in\mathbb{N}}$ is an increasing family of compact sets such that $\mathbb{R}^d = \bigcup_n X_n$.

Lemma 6. *We have*

$$\liminf_{\varepsilon\to 0^+} \liminf_{n\to\infty} I(n,\varepsilon) \geqslant \gamma \int_0^\infty \int_{\mathbb{R}^d} |F(x,t)|_* |G(x,t)|_*\, dx\, t\, dt.$$

Proof. In order to make formulae more compact, we will sometimes write ∂_{x_0} instead of ∂_t. The first step will be to prove that

$$\begin{aligned}\left(\partial_t^2 + \Delta\right)(b_\kappa)(x,t) \geqslant{}& \gamma |F(x,t)|_* |G(x,t)|_* - \kappa r(x) E_\kappa(u(x,t)) \\ &+ r(x) B_\kappa(u(x,t)) - 2\sum_{i=1}^d \partial_{\eta_i} B_\kappa(u(x,t)) Q_t^i g_i(x).\end{aligned} \tag{3.1}$$

From the chain rule we get $\partial_{x_i} b_\kappa(x,t) = \langle \nabla B_\kappa(u(x,t)), \partial_{x_i} u(x,t)\rangle$ for $i = 0,\dots,d$. Then, again by the chain rule, we have

$$\partial_{x_i}^2 b_\kappa(x,t) = \left\langle \nabla B_\kappa(u(x,t)), \partial_{x_i}^2 u(x,t)\right\rangle + \langle \mathrm{Hess}(B_\kappa)(u(x,t))(\partial_{x_i} u(x,t)), \partial_{x_i} u(x,t)\rangle.$$

Summing for $i = 0,\dots,d$, we get

$$\begin{aligned}\left(\partial_t^2 + \Delta\right)(b_\kappa)(x,t) ={}& \left\langle \nabla B_\kappa(u(x,t)), (\partial_t^2 + \Delta)(u)(x,t)\right\rangle \\ &+ \sum_{i=0}^d \langle \mathrm{Hess}(B_\kappa)(u(x,t))(\partial_{x_i} u(x,t)), \partial_{x_i} u(x,t)\rangle.\end{aligned}$$

By the definition of P_t and Q_t we see that

$$(\partial_t^2 - L')P_t f = 0$$

and

$$(\partial_t^2 - L')Q_t^i g_i = (\partial_t^2 - M_i)Q_t^i g_i - 2Q_t^i g_i = -2Q_t^i g_i.$$

Therefore, using the fact that $-\Delta = L' - r$ we get

$$\begin{aligned}\left(\partial_t^2+\Delta\right)(b_\kappa)(x,t) = r(x)\left\langle \nabla B_\kappa(u(x,t)), u(x,t)\right\rangle \\ - 2\sum_{i=1}^{d} \partial_{\eta_i} B_\kappa(u(x,t)) Q_t^i g_i(x) \\ + \sum_{i=0}^{d} \left\langle \mathrm{Hess}(B_\kappa)(u(x,t))\left(\partial_{x_i} u(x,t)\right), \partial_{x_i} u(x,t)\right\rangle .\end{aligned}$$

Next, inequalities 3. and 4. from Theorem 5 and the inequality of arithmetic and geometric means imply that

$$\begin{aligned}\left(\partial_t^2+\Delta\right)(b_\kappa)(x,t) \geqslant\, & r(x)\frac{\gamma}{2}\left(\tau_\kappa |P_t f(x)|^2 + {\tau_\kappa}^{-1}|Q_t g(x)|^2\right) \\ & - r(x)\kappa E_\kappa(u(x,t)) + r(x) B_\kappa(u(x,t)) \\ & - 2\sum_{i=1}^{d} \partial_{\eta_i} B_\kappa(u(x,t)) Q_t^i g_i(x) \\ & + \frac{\gamma}{2}\sum_{i=0}^{d}\left(\tau_\kappa |\partial_{x_i} P_t f(x)|^2 + {\tau_\kappa}^{-1}|\partial_{x_i} Q_t g(x)|^2\right) \\ =\, & \frac{\gamma\tau_\kappa |P_t f(x)|_*^2 + \gamma{\tau_\kappa}^{-1}|Q_t g(x)|_*^2}{2} - r(x)\kappa E_\kappa(u(x,t)) \\ & + r(x) B_\kappa(u(x,t)) - 2\sum_{i=1}^{d} \partial_{\eta_i} B_\kappa(u(x,t)) Q_t^i g_i(x) \\ \geqslant\, & \gamma |F(x,t)|_* |G(x,t)|_* - \kappa r(x) E_\kappa(u(x,t)) \\ & + r(x) B_\kappa(u(x,t)) - 2\sum_{i=1}^{d} \partial_{\eta_i} B_\kappa(u(x,t)) Q_t^i g_i(x).\end{aligned}$$

In summary

$$\begin{aligned}\left(\partial_t^2+\Delta\right)(b_\kappa)(x,t) \geqslant\, & \gamma |F(x,t)|_* |G(x,t)|_* - \kappa r(x) E_\kappa(u(x,t)) \\ & + r(x) B_\kappa(u(x,t)) - 2\sum_{i=1}^{d} \partial_{\eta_i} B_\kappa(u(x,t)) Q_t^i g_i(x).\end{aligned} \tag{3.2}$$

The next step is to show that

$$r(x) B(u(x,t)) - 2\sum_{i=1}^{d} \partial_{\eta_i} B(u(x,t)) Q_t^i g_i(x) \geqslant 0. \tag{3.3}$$

We have the following equalities:

$$\frac{\partial \beta}{\partial y}(x,y) = qy^{q-1} + \gamma \begin{cases} (2-q)x^2y^{1-q} \\ (2-q)y^{q-1} \end{cases},$$

$$\frac{\partial|\eta|}{\partial \eta_i} = \frac{\partial\sqrt{\eta_1^2+\cdots+\eta_d^2}}{\partial \eta_i} = \frac{\eta_i}{\sqrt{\eta_1^2+\cdots+\eta_d^2}} = \frac{\eta_i}{|\eta|},$$

$$\begin{aligned} 2\frac{\partial}{\partial \eta_i}B(\zeta,\eta) &= \frac{\partial}{\partial \eta_i}\beta(|\zeta|,|\eta|) = \frac{\partial \beta}{\partial y}(|\zeta|,|\eta|)\cdot\frac{\partial|\eta|}{\partial \eta_i} \\ &= \left(q|\eta|^{q-1} + \gamma(2-q)\begin{cases} |\zeta|^2|\eta|^{1-q} \\ |\eta|^{q-1} \end{cases}\right)\frac{\eta_i}{|\eta|}. \end{aligned}$$

Using them, we may rewrite inequality (3.3) as

$$\begin{aligned} &\left(|x|^2+2d\right)\left(|\zeta|^p + |\eta|^q + \gamma\begin{cases} |\zeta|^2|\eta|^{2-q} \\ \frac{2}{p}|\zeta|^p + \left(\frac{2}{q}-1\right)|\eta|^q \end{cases}\right) - \\ &\qquad 2\left(q|\eta|^q + \gamma(2-q)\begin{cases} |\zeta|^2|\eta|^{2-q} \\ |\eta|^q \end{cases}\right) \geqslant 0, \end{aligned} \tag{3.4}$$

where $\zeta = P_tf(x)$ and $\eta = Q_tg(x)$. Then, we consider two cases.
Case 1: $|\zeta|^p \leqslant |\eta|^q$. We omit $|x|^2$ reducing (3.4) to

$$d|\zeta|^p + (d-q)|\eta|^q + \gamma(d-2+q)|\zeta|^2|\eta|^{2-q} \geqslant 0.$$

Since $q \leqslant 2$, this is true as long as $d \geqslant 2$.
Case 2: $|\zeta|^p \geqslant |\eta|^q$. In this case inequality (3.4) becomes

$$\left(|x|^2+2d\right)\left(1+\frac{2\gamma}{p}\right)|\zeta|^p + \left(\left(|x|^2+2d\right)\left(1+\frac{2\gamma}{q}-\gamma\right) - 2q - 2\gamma(2-q)\right)|\eta|^q \geqslant 0.$$

We omit the first term, $|x|^2$ and $|\eta|^q$ in the above. Then we are left with proving

$$2d\left(1+\frac{2\gamma}{q}-\gamma\right) - 2q - 4\gamma + 2\gamma q \geqslant 0.$$

Plugging the definition of γ into this inequality and rearranging it, we arrive at

$$q^3 + q^2(-d-3) + q(3d-6) + 6d \geqslant 0,$$

which is true for $1 < q \leqslant 2$ and $d \geqslant 2$.

Having proved (3.3), we come back to (3.2) and write

$$\begin{aligned}\left(\partial_t^2+\Delta\right)(b_\kappa)(x,t) \geqslant\ & \gamma|F(x,t)|_*|G(x,t)|_* - \kappa r(x)E_\kappa(u(x,t))\\ &+ r(x)B_\kappa(u(x,t)) - 2\sum_{i=1}^d \partial_{\eta_i}B_\kappa(u(x,t))Q_t^i g_i(x)\\ &- r(x)B(u(x,t)) + 2\sum_{i=1}^d \partial_{\eta_i}B(u(x,t))Q_t^i g_i(x).\end{aligned} \tag{3.5}$$

The last step is to show that

$$\kappa r(x)E_\kappa(u(x,t))$$

and the difference between

$$r(x)B(u(x,t)) - 2\sum_{i=1}^d \partial_{\eta_i}B(u(x,t))Q_t^i g_i(x)$$

and

$$r(x)B_\kappa(u(x,t)) - 2\sum_{i=1}^d \partial_{\eta_i}B_\kappa(u(x,t))Q_t^i g_i(x)$$

are negligible.

First let us prove that $u(x,t)$ is bounded on $X_n\times[0,+\infty)$. Recall that

$$u(x,t) = (P_tf(x), Q_tg(x)) = \left(P_tf(x), Q_t^1g_1(x),\dots,Q_t^dg_d(x)\right),$$

where

$$P_tf = \sum_{n\in\mathbb{N}^d} e^{-t\lambda_n'^{1/2}}\langle f,h_n\rangle h_n,\quad Q_t^ig_i = \sum_{n\in\mathbb{N}^d} e^{-t\lambda_n'^{1/2}}\left\langle g_i, c_n^i\delta_i^*h_n\right\rangle c_n^i\delta_i^*h_n$$

and $f, g_i \in \mathcal{D}$. Since h_k are continuous, they are bounded on X_n, thus

$$|P_tf(x)| \leqslant \sum_{k\in\mathbb{N}^d} e^{-t\lambda_k'^{1/2}}|\langle f,h_k\rangle|M_{n,k}$$

for some constants $M_{n,k}$. The above sum has only finitely many non-zero terms and it is a decreasing function of t, so $P_tf(x)$ is bounded uniformly for all $x\in X_n$ and $t\geqslant 0$. A similar argument shows that each $Q_t^ig_i$ is bounded.

Using inequality 5. from Theorem 5 and the previous paragraph, we see that there exists a sequence $\{\kappa(n)\}_{n\in\mathbb{N}}$ such that

$$\int_{X_n}\left|\kappa(n)r(x)E_{\kappa(n)}(u(x,t))\right|dx \leqslant \frac{1}{n}. \tag{3.6}$$

Now we turn to estimating $|B(u(x,t)) - B_\kappa(u(x,t))|$. As we have shown, $u[X_n \times [0,+\infty)]$ is bounded, which means that B is uniformly continuous on this set. Therefore, for each $n \in \mathbb{N}$ there exists $\kappa(n)$ satisfying (3.6) and such that for all $x \in X_n$ and $t \geqslant 0$

$$\begin{aligned}\left|B(u(x,t)) - B_{\kappa(n)}(u(x,t))\right| &\leqslant \int_{B(0,\kappa(n))} |B(u(x,t)) - B(u(x,t)-y)|\psi_{\kappa(n)}(y)\,dy \\ &\leqslant \frac{1}{n}\left(\int_{X_n} |r(x)|\,dx\right)^{-1}. \end{aligned} \tag{3.7}$$

A similar reasoning shows that for each $n \in \mathbb{N}$ there exists $\kappa(n)$ satisfying (3.6) and (3.7) and such that for all $x \in X_n$, $t \geqslant 0$ and $i = 1, \ldots, d$

$$\left|\partial_{\eta_i} B(u(x,t)) - \partial_{\eta_i} B_{\kappa(n)}(u(x,t))\right| \leqslant \frac{1}{n}\left(\int_{X_n} \left|2Q_t^i g_i(x)\right| dx\right)^{-1}. \tag{3.8}$$

Coming back to inequality (3.5), we get

$$\begin{aligned}&\int_{X_n} \left(\partial_t^2 + \Delta\right)(b_{\kappa(n)})(x,t)\,dx \\ &\quad\geqslant \gamma \int_{X_n} |F(x,t)|_* |G(x,t)|_*\,dx - \int_{X_n} \kappa(n) r(x) E_{\kappa(n)}(u(x,t))\,dx \\ &\quad+ \int_{X_n} r(x)\left(B_{\kappa(n)}(u(x,t)) - B(u(x,t))\right)\,dx \\ &\quad- 2\int_{X_n} \sum_{i=1}^d Q_t^i g_i(x)\left(\partial_{\eta_i} B_{\kappa(n)}(u(x,t)) - \partial_{\eta_i} B(u(x,t))\right)\,dx.\end{aligned}$$

Using conditions (3.6), (3.7) and (3.8) on $\kappa(n)$ we get

$$\liminf_{n\to\infty} \int_{X_n} \left(\partial_t^2 + \Delta\right)(b_{\kappa(n)})(x,t)\,dx \geqslant \gamma \int_{\mathbb{R}^d} |F(x,t)|_* |G(x,t)|_*\,dx$$

and by the monotone convergence theorem

$$\liminf_{\varepsilon\to 0^+} \liminf_{n\to\infty} I(n,\varepsilon) \geqslant \gamma \int_0^\infty \int_{\mathbb{R}^d} |F(x,t)|_* |G(x,t)|_*\,dx\,t\,dt.$$

□

Lemma 7. *For $f, g \in \mathcal{D}$ we have*

$$\limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} I(n,\varepsilon) \geqslant \frac{1+\gamma}{2}\left(\|f\|_p^p + \|g\|_q^q\right).$$

Proof. Denote

$$I_1(n,\varepsilon) = \int_0^\infty \int_{X_n} \partial_t^2 \left(b_{\kappa(n)}\right)(x,t)\,dx\,te^{-\varepsilon t}\,dt,$$

$$I_2(n,\varepsilon) = \int_0^\infty \int_{X_n} \Delta \left(b_{\kappa(n)}\right)(x,t)\,dx\,te^{-\varepsilon t}\,dt.$$

Then $I(n,\varepsilon) = I_1(n,\varepsilon) + I_2(n,\varepsilon)$. First we prove that $\lim_{n\to\infty} I_2(n,\varepsilon) = 0$. Since

$$I_2(n,\varepsilon) = \sum_{i=1}^d \int_0^\infty \int_{X_n} \partial_{x_i}^2 \left(b_{\kappa(n)}\right)(x,t)\,dx\,te^{-\varepsilon t}\,dt,$$

it is sufficient to prove that each summand tends to 0. We will present the proof for the first term only, call it $I_2^1(n,\varepsilon)$. Let $x' = (x_2,\ldots,x_d)$. Integrating by parts with respect to x_1, we get

$$I_2^1(n,\varepsilon) = \int_0^\infty \int_{[-n,n]^{d-1}} \partial_{x_1}\left(b_{\kappa(n)}\right)(n,x',t) - \partial_{x_1}\left(b_{\kappa(n)}\right)(-n,x',t)\,dx'\,te^{-\varepsilon t}\,dt.$$

By the chain rule

$$\begin{aligned}\partial_{x_1}\left(b_{\kappa(n)}\right)(\pm n,x',t) &= \partial_\zeta B_{\kappa(n)}(u(\pm n,x',t))\partial_{x_1}P_tf(\pm n,x')\\ &\quad + \left\langle \nabla_\eta B_{\kappa(n)}(u(\pm n,x',t)), \partial_{x_1}Q_tg(\pm n,x')\right\rangle.\end{aligned}$$

Recall that $f, g_i \in \mathcal{D}$ and hence $P_tf, Q_t^i g_i \in \mathcal{D}$. Using item 2. of Theorem 5 and the fact that the Hermite functions converge to 0 rapidly we conclude that $\lim_{n\to\infty} I_2(n,\varepsilon) = 0$.

Now we turn to I_1. Using Fubini's theorem, we may interchange the order of integration to get

$$I_1(n,\varepsilon) = \int_{X_n} \int_0^\infty \partial_t^2 \left(b_{\kappa(n)}\right)(x,t)\,te^{-\varepsilon t}\,dt\,dx.$$

Next, we use integration by parts on the inner integral twice, neglecting the boundary terms (this is allowed by the same argument as in the previous paragraph). This leads to

$$\begin{aligned}I_1(n,\varepsilon) &= -\int_{X_n} \int_0^\infty \partial_t \left(b_{\kappa(n)}\right)(x,t)\,(1-\varepsilon t)e^{-\varepsilon t}\,dt\,dx\\ &= \int_{X_n} b_{\kappa(n)}(x,0)\,dx + \varepsilon^2 \int_{X_n}\int_0^\infty b_{\kappa(n)}(x,t)\,te^{-\varepsilon t}\,dt\,dx\\ &\quad - 2\varepsilon \int_{X_n}\int_0^\infty b_{\kappa(n)}(x,t)\,e^{-\varepsilon t}\,dt\,dx\\ &\leqslant \int_{X_n} b_{\kappa(n)}(x,0)\,dx + \varepsilon^2 \int_{X_n}\int_0^\infty b_{\kappa(n)}(x,t)\,te^{-\varepsilon t}\,dt\,dx.\end{aligned}$$

Denote the last two terms by $I_1^1(n)$ and $I_1^2(n,\varepsilon)$.

First we will show that $\limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty} I_1^2(n,\varepsilon)=0$. Item 1. of Theorem 5 implies that

$$I_1^2(n,\varepsilon)\leqslant \varepsilon^2 C_p\int_{X_n}\int_0^\infty \left(|P_tf(x)|^p+|Q_tg(x)|^q+\max\left(\kappa(n)^p,\kappa(n)^q\right)\right)\, te^{-\varepsilon t}\,dt\,dx.$$

Taking $\kappa(n)$ satisfying (3.6), (3.7) and (3.8) and such that

$$(2n)^d\max\left(\kappa(n)^p,\kappa(n)^q\right)\leqslant \frac{1}{n}, \tag{3.9}$$

we get

$$\limsup_{n\to\infty} I_1^2(n,\varepsilon)\leqslant \varepsilon^2 C_p\int_X\int_0^\infty \left(|P_tf(x)|^p+|Q_tg(x)|^q\right)\, t\,dt\,dx\leqslant C\varepsilon^2.$$

The last step is to estimate $I_1^1(n)$. Using item 1. of Theorem 5 again, we obtain

$$I_1^1(n)\leqslant \frac{1+\gamma}{2}\int_{X_n}\left(|f(x)|+\kappa(n)\right)^p\,dx+\frac{1+\gamma}{2}\int_{X_n}\left(|g(x)|+\kappa(n)\right)^q\,dx.$$

We take $\varepsilon>0$, denote $A=\{x\in\mathbb{R}^d:\varepsilon|f(x)|\geqslant|\kappa(n)|\}$ and split these two integrals as follows:

$$\begin{aligned}
I_1^1(n)&\leqslant \frac{1+\gamma}{2}\int_A\left(|f(x)|+\kappa(n)\right)^p\,dx+\int_{A^{\mathrm{C}}}\left(|f(x)|+\kappa(n)\right)^p\,dx\\
&\quad+\frac{1+\gamma}{2}\int_A\left(|g(x)|+\kappa(n)\right)^q\,dx+\int_{A^{\mathrm{C}}}\left(|g(x)|+\kappa(n)\right)^q\,dx\\
&\leqslant \frac{1+\gamma}{2}\left((1+\varepsilon)^p\|f\|_p^p+(1+\varepsilon)^q\|g\|_q^q\right)\\
&\quad+\frac{1+\gamma}{2}(2n)^d\left(\left(1+\varepsilon^{-1}\right)^p\kappa(n)^p+\left(1+\varepsilon^{-1}\right)^q\kappa(n)^q\right).
\end{aligned}$$

Since $\kappa(n)$ satisfies (3.9), we get

$$\limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty} I_1^1(n,\varepsilon)\leqslant \frac{1+\gamma}{2}\left(\|f\|_p^p+\|g\|_q^q\right)$$

and hence, as we have shown that other terms are negligible, we obtain

$$\limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty} I(n,\varepsilon)\leqslant \frac{1+\gamma}{2}\left(\|f\|_p^p+\|g\|_q^q\right).$$

□

Now we are ready to prove the bilinear embedding theorem.

Proof of Theorem 4. Combining Lemma 6 and Lemma 7, we get

$$\int_0^\infty \int_{\mathbb{R}^d} |F(x,t)|_* |G(x,t)|_* \, dx \, t \, dt \leqslant \frac{1+\gamma}{2\gamma} \left(\|f\|_p^p + \|g\|_q^q \right).$$

Multiplying f by $\left(\frac{q\|g\|_q^q}{p\|f\|_p^p}\right)^{\frac{1}{p+q}}$ and g by the reciprocal of this number, we obtain

$$\int_0^\infty \int_{\mathbb{R}^d} |F(x,t)|_* |G(x,t)|_* \, dx \, t \, dt \leqslant \frac{1+\gamma}{2\gamma} \left(\left(\frac{q}{p}\right)^{1/q} + \left(\frac{p}{q}\right)^{1/p} \right) \|f\|_p \|g\|_q.$$

We need to show that $\frac{1+\gamma}{2\gamma}\left(\left(\frac{q}{p}\right)^{1/q} + \left(\frac{p}{q}\right)^{1/p}\right) \leqslant 6(p^*-1)$. Recall that $p \geqslant 2$, so $p^* = p$ and $1 < q \leqslant 2$, hence

$$\begin{aligned} \frac{1+\gamma}{2\gamma} \left(\left(\frac{q}{p}\right)^{1/q} + \left(\frac{p}{q}\right)^{1/p} \right) &= \frac{8+q(q-1)}{2} (q-1)^{\frac{1}{q}-1}(p-1) \\ &\leqslant (q+3)(q-1)^{\frac{1}{q}-1}(p-1) \leqslant 6(p-1). \end{aligned}$$

A proof of the last inequality can be found in [16, pp. 15–16]. If $p \leqslant 2$, we swap p with q and $P_t f$ with $Q_t g$ in the definition of b_κ, i.e., it becomes $b_\kappa(x,t) = B_\kappa(Q_t g(x), P_t f(x))$, and we proceed as before. Since $p^* = \max(p,q)$, the conclusion holds. ☐

3.3. **Proof of Theorem 2.** Having proved the bilinear embedding theorem, we move on to the main result of this section.

Proof. If $d = 1$, then, by (2.1), $L' = L + 2$ and equations (4.8) and (4.9) imply that $\mathbf{R}'$ is the adjoint of $\mathbf{R}$ from Section 5.4 of [16], so Theorem 10 (there) gives the desired result. Now assume that $d \geqslant 2$. By duality, it is sufficient to prove that

$$\left| \sum_{i=1}^d \langle R_i' f, g_i \rangle \right| \leqslant 48(p^*-1)\|f\|_p \left\| \left(\sum_{i=1}^d |g_i|^2 \right)^{1/2} \right\|_q$$

for any $f, g_i \in \mathcal{D}$. Since $\mathcal{D}$ is dense in L^p for $1 \leqslant p < \infty$, this will mean that $\mathbf{R}'$ admits a bounded extension to the whole L^p space with the same norm. By Lemma 3, we have

$$\begin{aligned}
\left|\sum_{i=1}^{d}\langle R_i' f, g_i\rangle\right| &\leqslant 4\int_0^\infty \sum_{i=1}^{d}\left|\left\langle \delta_i^* P_t f, \partial_t Q_t^i g\right\rangle\right| t\,dt \\
&\leqslant 4\int_0^\infty \int_{\mathbb{R}^d}\sum_{i=1}^{d}\left(|\partial_{x_i}P_t f(x)| + |x_i P_t f(x)|\right)\left|\partial_t Q_t^i g_i(x)\right| dx\,t\,dt \\
&\leqslant 4\int_0^\infty \int_{\mathbb{R}^d}\left(\left(\sum_{i=1}^{d}|\partial_{x_i}P_t f(x)|^2\right)^{1/2} + \sqrt{r(x)}|P_t f(x)|\right)|G(x,t)|_*\,dx\,t\,dt \\
&\leqslant 8\int_0^\infty \int_{\mathbb{R}^d}|F(x,t)|_*|G(x,t)|_*\,dx\,t\,dt \leqslant 48(p^*-1)\|f\|_p\left\|\left(\sum_{i=1}^{d}|g_i|^2\right)^{1/2}\right\|_q.
\end{aligned}$$

The last inequality follows from Theorem 4. □

4. Riesz transforms of the second kind

This section is devoted to estimating the norm of the vector of the Riesz transforms

$$\tilde{R}_i f(x) = \delta_i^* L^{-1/2} f(x).$$

As noted earlier, we will give a result similar to Corollary 1 from [5] but with an explicit constant.

We want to estimate

$$\left\|\tilde{\mathbf{R}}f\right\|_p := \left(\int_{\mathbb{R}^d}\left|\tilde{\mathbf{R}}f(x)\right|^p dx\right)^{1/p}.$$

Observe that for $f \in \mathcal{D}$ it holds

$$\begin{aligned}
\tilde{R}_i f(x) &= \delta_i^* L^{-1/2} f(x) = \left(-\partial_{x_i} + x_i\right) L^{-1/2} f(x) \\
&= -\delta_i L^{-1/2} f(x) + 2x_i L^{-1/2} f(x) \\
&= R_i^1 f(x) + R_i^2 f(x).
\end{aligned}$$

Then $\tilde{\mathbf{R}}f(x) = \mathbf{R^1}f(x) + \mathbf{R^2}f(x)$ (with $\tilde{\mathbf{R}}f(x) = \left(\tilde{R}_1 f(x), \ldots, \tilde{R}_d f(x)\right)$ and $\mathbf{R^1}$ and $\mathbf{R^2}$ defined analogously), hence

$$\left|\tilde{\mathbf{R}}f(x)\right| \leqslant \left|\mathbf{R^1}f(x)\right| + \left|\mathbf{R^2}f(x)\right|$$

and

$$\left\|\tilde{\mathbf{R}}f\right\|_p \leqslant \left\|\mathbf{R^1}f\right\|_p + \left\|\mathbf{R^2}f\right\|_p. \tag{4.1}$$

Theorem 10 from [16] gives the bound of $48(p^*-1)$ for the L^p norm of $\mathbf{R^1}$, so we will focus on $\mathbf{R^2}$. Next, note that

$$\left|\mathbf{R^2} f(x)\right| = 2\left(\sum_{i=1}^{d}\left|x_i L^{-1/2} f(x)\right|^2\right)^{1/2} = 2|x|\left|L^{-1/2} f(x)\right|,$$

which means that it is sufficient to deal with the operator $|x|L^{-1/2}$, formally defined on $\mathcal{D}$ as $Sf(x) = |x|L^{-1/2}f(x)$. This operator turns out to be bounded on all L^p spaces for $1 \leqslant p < \infty$.

Theorem 8. *For $1 \leqslant p < \infty$ we have $\|S\|_{p\to p} \leqslant 3$.*

In order to prove this theorem, we first derive an expression for the kernel of S, i.e., a function $K(x,y)$ such that

$$Sf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\, dy \quad \text{for } f \in \mathcal{D}.$$

Lemma 9. *For $x, y \in \mathbb{R}^d$ we have*

$$K(x,y) = |x| \int_0^\infty \frac{1}{\sqrt{t}} K_t(x,y)\, dt,$$

where

$$K_t(x,y) = \frac{C_d}{(\sinh 2t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4}|x+y|^2\right), \quad C_d = \frac{1}{(2\pi)^{d/2}\sqrt{\pi}}.$$

Proof. Equation (16) in [6] states that

$$e^{-tL} f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} K_t'(x,y) f(y)\, dy,$$

with

$$\begin{aligned} K_t'(x,y) &= \frac{1}{(\sinh 2t)^{d/2}} \exp\left(-\frac{|x|^2+|y|^2}{2}\coth 2t + \frac{\langle x,y\rangle}{\sinh 2t}\right) \\ &= \frac{1}{(\sinh 2t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4}|x+y|^2\right). \end{aligned}$$

Note also that

$$\lambda^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t\lambda} \frac{1}{\sqrt{t}}\, dt.$$

Since $\mathcal{D} = \text{lin}\{h_n : n \in \mathbb{N}^d\}$, it is sufficient to prove the formula for $f = h_n$. We have

$$\begin{aligned} L^{-1/2}h_n(x) = \lambda_n^{-1/2}h_n(x) &= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t\lambda_n}h_n(x)\frac{1}{\sqrt{t}}\,dt \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-tL}h_n(x)\frac{1}{\sqrt{t}}\,dt \\ &= \frac{1}{\sqrt{\pi}}\frac{1}{(2\pi)^{d/2}}\int_0^\infty \frac{1}{\sqrt{t}}\int_{\mathbb{R}^d} K_t'(x,y)h_n(y)\,dy\,dt. \end{aligned}$$

This integral is absolutely convergent, so we may interchange the order of integration and the conclusion follows. ☐

Next we prove that the operator T defined on L^p, $1 \leqslant p \leqslant \infty$, as

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,dy$$

is bounded uniformly in d and p. This will mean that S is bounded on $\mathcal{D}$ in L^p norm and, by density, that it has a unique bounded extension to L^p for $1 \leqslant p < \infty$ with the same norm. We want to use interpolation and our goal is to prove that

$$\int_{\mathbb{R}^d} K(x,z)\,dz \leqslant 2 \quad \text{and} \quad \int_{\mathbb{R}^d} K(z,y)\,dz \leqslant 3 \tag{4.2}$$

for all $x, y \in \mathbb{R}^d$. Clearly, we have

$$\begin{aligned} \int_{\mathbb{R}^d} K(z,y)\,dz &= \int_{\mathbb{R}^d} |z| \int_0^\infty \frac{1}{\sqrt{t}}K_t(z,y)\,dt\,dz \\ &\leqslant \int_{\mathbb{R}^d} |y| \int_0^\infty \frac{1}{\sqrt{t}}K_t(z,y)\,dt\,dz \\ &\quad + \int_{\mathbb{R}^d} |y-z| \int_0^\infty \frac{1}{\sqrt{t}}K_t(z,y)\,dt\,dz, \end{aligned} \tag{4.3}$$

so, by symmetry of K_t, it is sufficient to prove the first inequality of (4.2) and the following proposition.

Proposition 10. *For $y \in \mathbb{R}^d$ it holds*

$$\int_{\mathbb{R}^d} |y-z| \int_0^\infty \frac{1}{\sqrt{t}}K_t(z,y)\,dt\,dz \leqslant 1. \tag{4.4}$$

Proof. We begin with an auxiliary computation:

$$I(k) := \int_{\mathbb{R}^d} |x|e^{-k|x|^2}\,dx = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\pi^{d/2}}{k^{(d+1)/2}} \quad \text{for } k \geqslant 0. \tag{4.5}$$

To prove (4.5), let $S_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$ denote the surface area of the unit sphere in the d-dimensional Euclidean space. Then we can write

$$\begin{aligned}\int_{\mathbb{R}^d} |x| e^{-k|x|^2}\, dx &= \int_0^\infty r e^{-kr^2} r^{d-1} S_d\, dr = \frac{S_d}{2k^{(d+1)/2}} \int_0^\infty x^{(d-1)/2} e^{-x}\, dx \\ &= \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{\pi^{d/2}}{k^{(d+1)/2}}.\end{aligned}$$

Coming back to (4.4), in view of (4.5) we have, for $t \geqslant 0$,

$$\begin{aligned}\int_{\mathbb{R}^d} |x-y| K_t(x,y)\, dx &= \frac{C_d}{(\sinh 2t)^{d/2}} \int_{\mathbb{R}^d} |x-y| \exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4}|x+y|^2\right) dx \\ &\leqslant \frac{C_d}{(\sinh 2t)^{d/2}} \int_{\mathbb{R}^d} |x-y| \exp\left(-\frac{|x-y|^2}{4\tanh t}\right) dx \\ &= \frac{C_d}{(\sinh 2t)^{d/2}} \int_{\mathbb{R}^d} |x| \exp\left(-\frac{|x|^2}{4\tanh t}\right) dx \\ &= \frac{C_d}{(\sinh 2t)^{d/2}} I\left(\frac{1}{4\tanh t}\right) \\ &= \frac{\pi^{d/2}}{(2\pi)^{d/2}\sqrt{\pi}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{(4\tanh t)^{(d+1)/2}}{(\sinh 2t)^{d/2}} \\ &= \frac{1}{2^{d/2}\sqrt{\pi}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \frac{(4\tanh t)^{(d+1)/2}}{(\sinh 2t)^{d/2}}.\end{aligned}$$

Plugging it into (4.4), we get

$$\int_{\mathbb{R}^d} |y-z| \int_0^\infty \frac{1}{\sqrt{t}} K_t(z,y)\, dt\, dz \leqslant \frac{1}{2^{d/2}\sqrt{\pi}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \frac{(4\tanh t)^{(d+1)/2}}{(\sinh 2t)^{d/2}} \frac{dt}{\sqrt{t}}.$$

To estimate the last integral, we will use formula [12, 5.12.7]:

$$\int_0^\infty \frac{1}{(\cosh t)^{2a}} dt = 4^{a-1} \mathrm{B}(a,a),$$

where B denotes the beta function. We obtain

$$\begin{aligned}\int_0^\infty \frac{(4\tanh t)^{(d+1)/2}}{(\sinh 2t)^{d/2}}\frac{dt}{\sqrt{t}} &= \frac{4^{(d+1)/2}}{2^{d/2}}\int_0^\infty \left(\frac{\tanh t}{t}\right)^{1/2}\frac{1}{(\cosh t)^d}\,dt\\ &\leqslant 2^{\frac{d}{2}+1}\int_0^\infty \frac{1}{(\cosh t)^d}\,dt = 2^{\frac{d}{2}+1}\cdot 4^{\frac{d}{2}-1}\mathrm{B}\left(\frac{d}{2},\frac{d}{2}\right)\\ &= 2^{\frac{3d}{2}-1}\frac{\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)}.\end{aligned}$$

Finally, using the Legendre duplication formula ($\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$), we get

$$\begin{aligned}&\int_{\mathbb{R}^d}|y-z|\int_0^\infty \frac{1}{\sqrt{t}}K_t(z,y)\,dt\,dz\\ &\leqslant 2^{\frac{3d}{2}-1}\frac{1}{2^{d/2}\sqrt{\pi}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} = 2^{d-1}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma(d)} = 1.\end{aligned}$$

□

Now it remains to justify the first inequality of (4.2).

Proposition 11. *For $x \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d}|x|\int_0^\infty \frac{1}{\sqrt{t}}K_t(x,y)\,dt\,dy \leqslant \frac{1}{\sqrt{\pi}} + \sqrt{2}.$$

Proof. The first step is to compute the integral $\int_{\mathbb{R}^d} K_t(x,y)\,dy$. Observe that

$$\begin{aligned}&\exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4}|x+y|^2\right) =\\ &\exp\left(-\frac{1}{4}\left|y\sqrt{\tanh t + \frac{1}{\tanh t}} + x\frac{\tanh t - \frac{1}{\tanh t}}{\sqrt{\tanh t + \frac{1}{\tanh t}}}\right|^2 - \frac{|x|^2}{\tanh t + \frac{1}{\tanh t}}\right) =\\ &\exp\left(-\frac{1}{4}\left|y\sqrt{2\coth(2t)} + x\frac{\tanh t - \frac{1}{\tanh t}}{\sqrt{\tanh t + \frac{1}{\tanh t}}}\right|^2 - \frac{|x|^2}{2\coth(2t)}\right),\end{aligned}$$

hence

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4}|x+y|^2\right) dy = \\
&\exp\left(-\frac{|x|^2}{2\coth(2t)}\right) \int_{\mathbb{R}^d} \exp\left(-\frac{1}{4}\left|y\sqrt{2\coth(2t)}\right|^2\right) dy = \\
&\exp\left(-\frac{|x|^2}{2\coth(2t)}\right) \left(\frac{4\pi}{2\coth(2t)}\right)^{d/2},
\end{aligned}
$$

so that

$$
\int_{\mathbb{R}^d} K_t(x,y)\, dy = \frac{C_d}{(\sinh 2t)^{d/2}} \exp\left(-\frac{|x|^2}{2\coth(2t)}\right) \left(\frac{4\pi}{2\coth(2t)}\right)^{d/2}.
$$

To estimate the integral with respect to t, we need to split it into two parts. Note that for $t \geqslant 0$, $\frac{1}{t} \leqslant 2\coth(2t)$. Let $\tau \in [0.95, 0.96]$ denote the unique positive solution of $2\coth(2t) = \frac{2}{t}$. It follows that $2\coth(2t) \leqslant \frac{2}{t}$ for $0 \leqslant t \leqslant \tau$. Thus, we obtain

$$
\begin{aligned}
&|x| \int_0^\tau \frac{1}{(\sinh 2t)^{d/2}} \frac{1}{\sqrt{t}} \exp\left(-\frac{|x|^2}{2\coth(2t)}\right) \left(\frac{4\pi}{2\coth(2t)}\right)^{d/2} dt \leqslant \\
&|x| \int_0^\tau \frac{1}{(2t)^{d/2}} \exp\left(-\frac{t|x|^2}{2}\right) (4\pi)^{d/2} \frac{t^{d/2}}{\sqrt{t}}\, dt = \\
&|x|\,(2\pi)^{d/2} \int_0^\tau \exp\left(-\frac{t|x|^2}{2}\right) \frac{1}{\sqrt{t}}\, dt \leqslant \\
&|x|\,(2\pi)^{d/2} \int_0^\infty \exp\left(-\frac{t|x|^2}{2}\right) \frac{1}{\sqrt{t}}\, dt = |x|\,(2\pi)^{d/2} \sqrt{\frac{2\pi}{|x|^2}} = (2\pi)^{(d+1)/2}.
\end{aligned}
\tag{4.6}
$$

For the second part, when $t \geqslant \tau$ and $2\coth(2t) \leqslant \frac{2}{\tau}$, calculations are as follows:

$$\begin{aligned}
&|x| \int_\tau^\infty \frac{1}{(\sinh 2t)^{d/2}} \frac{1}{\sqrt{t}} \exp\left(-\frac{|x|^2}{2\coth(2t)}\right) \left(\frac{4\pi}{2\coth(2t)}\right)^{d/2} dt \leqslant \\
&|x| \exp\left(-\frac{\tau |x|^2}{2}\right) \int_\tau^\infty \frac{1}{(\sinh 2t)^{d/2}} \left(\frac{4\pi}{2}\right)^{d/2} \frac{1}{\sqrt{t}}\, dt \leqslant \\
&|x| \exp\left(-\frac{\tau |x|^2}{2}\right) \frac{1}{\sqrt{\tau}} (2\pi)^{d/2} \int_\tau^\infty \left(\frac{4}{e^{2t}}\right)^{d/2} dt \leqslant \\
&\frac{1}{\tau\sqrt{e}} (2\pi)^{d/2}\, 2^d \int_\tau^\infty e^{-td}\, dt \leqslant (2\pi)^{d/2}\, 2^d \frac{e^{-\tau d}}{d} \leqslant (2\pi)^{d/2}.
\end{aligned} \tag{4.7}$$

In the second inequality we used the fact that $\sinh(2t) \geqslant \frac{e^{2t}}{4}$ for $t \geqslant \tau$. Combining (4.6) and (4.7) and recalling the definition of K_t completes the proof. □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 8. Proposition 10, Proposition 11 and (4.3) imply that

$$\int_{\mathbb{R}^d} K(x,z)\, dz \leqslant 3 \quad \text{and} \quad \int_{\mathbb{R}^d} K(z,y)\, dz \leqslant 3,$$

hence T is bounded on L^1 and L^∞ with norm at most 3. Using the Riesz–Thorin interpolation theorem we obtain $\|T\|_{p\to p} \leqslant 3$ for $1 \leqslant p \leqslant \infty$ and since $S = T$ on $\mathcal{D}$ — a dense subspace of L^p for $1 \leqslant p < \infty$ — S has a unique bounded extension to L^p with norm at most 3. □

Recollecting (4.1), we see that Theorem 8 and Theorem 10 from [16] imply an L^p norm estimate for $\tilde{\mathbf{R}} f = \left(\tilde{R}_1 f, \ldots, \tilde{R}_d f\right)$.

Theorem 12. *For $f \in L^p$ we have*

$$\left\|\tilde{\mathbf{R}} f\right\|_p = \left(\int_{\mathbb{R}^d} \left|\tilde{\mathbf{R}} f(x)\right|^p dx\right)^{1/p} \leqslant 54(p^* - 1)\|f\|_p.$$

As a corollary of the above result we will prove one more theorem. Let

$$\mathbf{R}^* f = (R_1^* f, \ldots, R_d^* f)$$

with

$$R_i^* f(x) = \delta_i^* (L+2)^{-1/2} f(x).$$

It is worth noting that each R_i^* is the adjoint of $R_i = \delta_i L^{-1/2}$ — the 'usual' Riesz–Hermite transform. To prove it, we check that $\langle h_n, R_i^* h_k\rangle = \langle R_i h_n, h_k\rangle$. For the left-hand side we use item 2. from Lemma 1.

$$\begin{aligned}\langle h_n, R_i^* h_k\rangle &= \left\langle h_n, \delta_i^*(L+2)^{-1/2} h_k\right\rangle = (\lambda_k+2)^{-1/2}\langle h_n, \delta^* h_k\rangle \\ &= \sqrt{2(k_i+1)}(\lambda_k+2)^{-1/2}\langle h_n, h_{k+e_i}\rangle \\ &= \begin{cases}\sqrt{\frac{2(k_i+1)}{2|k|_1+d+2}} & \text{if } n=k+e_i \\ 0 & \text{otherwise}\end{cases}.\end{aligned} \tag{4.8}$$

For the right-hand side we use item 1.

$$\begin{aligned}\langle R_i h_n, h_k\rangle &= \left\langle \delta_i L^{-1/2} h_n, h_k\right\rangle = \lambda_n^{-1/2}\langle \delta_i h_n, h_k\rangle \\ &= \sqrt{2n_i}\lambda_n^{-1/2}\langle h_{n-e_i}, h_k\rangle \\ &= \begin{cases}\sqrt{\frac{2n_i}{2|n|_1+d}} & \text{if } n-e_i=k \\ 0 & \text{otherwise}\end{cases}.\end{aligned} \tag{4.9}$$

Now we are ready to state the last theorem of this paper.

Theorem 13. *For $f \in L^p$ we have*

$$\|\mathbf{R}^* f\|_p = \left(\int_{\mathbb{R}^d} |\mathbf{R}^* f(x)|^p\, dx\right)^{1/p} \leqslant 108(p^*-1)\|f\|_p.$$

To prove this theorem, we perform a slightly more general calculation. For $a>0$ we define

$$U_a f(x) = \left(L(L+2a)^{-1}\right)^{1/2} f(x), \quad f \in \mathcal{D}.$$

Proposition 14. *For $1 \leqslant p < \infty$ we have $\|U_a\|_{p\to p} \leqslant 2$.*

Proof. We begin with a well-known fact: If A is a positive operator and $\|A\| \leqslant 1$, then

$$(I-A)^{1/2} = I - \sum_{n=1}^{\infty} c_n A^n, \tag{4.10}$$

where

$$c_n = \frac{(2n)!}{(n!)^2(2n-1)4^n} \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = 1.$$

Next, observe that

$$\left(L(L+2a)^{-1}\right)^{1/2} = \left(I - 2a(L+2a)^{-1}\right)^{1/2},$$

so, taking $A = 2a\,(L+2a)^{-1}$ in (4.10), we see that it is enough to prove that $\left\|(L+2a)^{-1}\right\|_{p\to p} \leqslant \frac{1}{2a}$. We proceed as in the proof of Theorem 8. First, we find the kernel of $(L+2a)^{-1}$, then prove its boundedness on L^1 and L^∞ and finally use interpolation.

A computation similar to the proof of Lemma 9 shows that

$$(L+2a)^{-1} f(x) = \int_{\mathbb{R}^d} \tilde{K}(x,y) f(y)\, dy \quad \text{for } f \in D,$$

where

$$\tilde{K}(x,y) = \int_0^\infty e^{-2at} \tilde{K}_t(x,y)\, dt$$

and

$$\tilde{K}_t(x,y) = \frac{\tilde{C}_d}{(\sinh 2t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4} |x+y|^2 \right), \quad \tilde{C}_d = \frac{1}{(2\pi)^{d/2}}.$$

Since this time the kernel is symmetric, we only prove that

$$\int_{\mathbb{R}^d} \tilde{K}(x,y)\, dy \leqslant \frac{1}{2a}.$$

Calculations are as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} \tilde{K}(x,y)\, dy &= \tilde{C}_d \int_{\mathbb{R}^d} \int_0^\infty \frac{e^{-2at}}{(\sinh 2t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4\tanh t} - \frac{\tanh t}{4} |x+y|^2 \right) dt\, dy \\
&\leqslant \tilde{C}_d \int_0^\infty \frac{e^{-2at}}{(\sinh 2t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4\tanh t} \right) dy\, dt \\
&= \tilde{C}_d \int_0^\infty \frac{e^{-2at}}{(\sinh 2t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|y|^2}{4\tanh t} \right) dy\, dt \\
&= \tilde{C}_d \int_0^\infty \frac{e^{-2at}}{(\sinh 2t)^{d/2}} (4\pi \tanh t)^{d/2}\, dt \\
&= \tilde{C}_d \int_0^\infty e^{-2at} \frac{(4\pi)^{d/2}}{2^{d/2}} \frac{1}{(\cosh t)^d}\, dt \\
&= \int_0^\infty \frac{e^{-2at}}{(\cosh t)^d}\, dt.
\end{aligned}$$

We split the last integral into two parts — from 0 to 1 and from 1 to ∞. The first part can be estimated by

$$\int_0^1 \frac{e^{-2at}}{(\cosh t)^d}\, dt \leqslant \int_0^1 e^{-2at}\, dt = \frac{1-e^{-2a}}{2a}$$

and the second one by

$$\begin{aligned}\int_1^\infty \frac{e^{-2at}}{(\cosh t)^d}\,dt &= 2^d \int_1^\infty \frac{e^{-2at}}{(e^t+e^{-t})^d}\,dt\\ &\leqslant 2^d \int_1^\infty e^{-2at}e^{-td}\,dt\\ &= 2^d \frac{e^{-2a-d}}{2a+d}.\end{aligned}$$

Adding, we get

$$\begin{aligned}\int_0^\infty \frac{e^{-2at}}{(\cosh t)^d}\,dt &\leqslant \frac{1-e^{-2a}}{2a} + 2^d\frac{e^{-2a-d}}{2a+d}\\ &\leqslant \frac{1+2^d e^{-d}e^{-2a}-e^{-2a}}{2a} < \frac{1}{2a}.\end{aligned}$$

This means that the operator V defined as

$$Vf(x) = \int_{\mathbb{R}^d} \tilde{K}(x,y)f(y)\,dy$$

is bounded on L^1 and L^∞ with norm at most $\frac{1}{2a}$ and the Riesz–Thorin interpolation theorem gives its boundedness on L^p for $1 \leqslant p \leqslant \infty$ with the same upper bound for the norm. Density of $\mathcal{D}$ implies that $(L+2a)^{-1}$ has a unique bounded extension to the whole L^p space, $1 \leqslant p < \infty$, with norm at most $\frac{1}{2a}$. Applying (4.10) with $A = 2a\,(L+2a)^{-1}$ completes the proof. □

This leads us to the proof of Theorem 13.

Proof of Theorem 13. It is sufficient to note that for $f \in \mathcal{D}$

$$R_i^* f = \delta_i^* \left(L+2\right)^{-1/2} f = \delta_i^* L^{-1/2} \left(L(L+2)^{-1}\right)^{1/2} f = \tilde{R}_i U_1 f.$$

Now Theorem 12 and Proposition 14 complete the proof. □

Finally, let us mention that in the light of (2.1), a very similar argument (with U_d instead of U_1) can be used to prove Theorem 2 with the constant equal to 108.

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