Word Equations: Sheet 3

Task 1 Show Hadamard inequality for a square matrix $N = (n_{i,j})_{i,j=1}^{k}$

$$\det(N) \leq \prod_{j=1}^{k} \sqrt{\sum_{i=1}^{k} n_{i,j}^2}.$$

Task 2 Show that if a maximal block a^ℓ occurs in the length-minimal solution of a word equation than ℓ is a visible length.
Hint: Looking at a^ℓ may be not enough, try looking at $ba^\ell c$.

Task 3 Show that the exponential bound on the length of a-blocks for length-minimal solutions is tight (the exact constant at the exponent is not tight, though).

Task 4 Show that we can use the a-presentation approach for the compression algorithm: we do not guess the lengths of the a-prefixes and suffixes, but denote them as variables and we write an appropriate system of linear equations.

Show that when the word equation can be encoded using m bits (in a natural encoding) then the constructed system has size $\mathcal{O}(m)$ bits. (We can encode the constants in unary).

Task 5 Show that we can verify the system of linear Diophantine equations in which all constants are encoded in unary in linear space (counted in bits).
Hint: Repeatedly guess the parity of sides of all equations and divide by 2. Do you see some connection with block compression, both ways.

Task 6 Using the bound on the size of the minimal solutions of integer programming show that the doubly exponential bound on the size of the length-minimal solution follows from our original algorithm (that uses the block compression variant from Task 3.)

Task 7 Show that the exponential bound on the exponent of periodicity (but not with a $2^{\mathcal{O}(n)}$ bound, though) can be inferred already from our original algorithm for word equations plus the bound on the length of a-blocks in the length-minimal solutions.
Hint: How does the exponent of periodicity changes after one compression step? What is the difference between pair compression and block compression?

Task 8 The $\exists^*$-theory of word equations consists of all sentences of the form:

$$\exists_{x_1,x_2,\ldots,x_k} \varphi(x_1, x_2, \ldots, x_k)$$

where φ is quantifier-free logic formula that uses $\wedge, \vee, \neg$ as connectives and atomic formulas that are word equations that use constants from Σ^* and variables $x_1, x_2, \ldots, x_k$.

Show that we can verify sentences form this theory in PSPACE.
Hint: The algorithm will heavily employ non-determinism to reduce this case to a system of word equations. The inequalities are easy to handle: look for first differences.

Task 9 Long, probably as a seminar talk for 1 hour. 3 points? Using papers supplied on the webpage, show an exponential bound on the size of the smallest solution of the integer programming. Show such a bound for each minimal solution.