Word Equations: Sheet 9

Task 1 Given a word equation

$$A_0 X A_1 \dots A_{n_{\mathcal{A}}-1} X A_{n_{\mathcal{A}}} = X B_1 \dots B_{n_{\mathcal{B}}-1} X B_{n_{\mathcal{B}}} \ , \tag{1}$$

show that if the number of occurrences of variables on both sides are different than this equation has at most one solution and it can be easily given in linear time.

Task 2 Show that if in (1) the $A_{n_{\mathcal{A}}} \neq \epsilon$ and $B_{n_{\mathcal{B}}} = \epsilon$ then it has an equivalent equation in which $A'_{n_{\mathcal{A}}} = \epsilon$ and $B'_{n_{\mathcal{B}}} \neq \epsilon$.
Hint: [1]

Task 3 Show that for every solution S of a word equation such that $S(X) \neq \epsilon$ the first letter of $S(X)$ is the first letter of A_0 and the last the last letter of $A_{n_{\mathcal{A}}}$ or $B_{n_{\mathcal{B}}}$ (whichever is non-empty).

If $A_0 \in a^+$ then $S(X) \in a^+$ for each solution S of $\mathcal{A} = \mathcal{B}$.

If the first letter of A_0 is a and $A_0 \notin a^+$ then there is at most one solution $S(X) \in a^+$, existence of such a solution can be tested (and its length returned) in $\mathcal{O}(|\mathcal{A}| + |\mathcal{B}|)$ time. Furthermore, for $S(X) \notin a^+$ the lengths of the a-prefixes of $S(X)$ and A_0 are the same.

Task 4 Let $A_0 \in a^+$. Show how to compute all solutions of the equation in linear time.
Hint: [2]

Task 5 Show that a word equation with 1 variable of length n has $\mathcal{O}(\log n)$ solutions and at most one infinite family of solutions of the form $\{w^k w' \ : \ k \geq 0\}$ and w' is a prefix of w.

Task 6 Show that the algorithm for quadratic equations in fact yields a description of all solutions of such na equation.

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[1] First show that there is a system of equivalent equations and then concatenate them.

[2] Look for the first non-a symbol at both sides of the equation and recurse on the rest.