Word Equations: Sheet 10

Task 1 Show that a positive theory of word equations over free semigroup is undecidable. Two alternations of quantifiers are enough (one, if you put some thought into it).
Hint: [1]

Task 2 Show that for large enough r_i there is a set of enough random string.
Hint: [2]

Task 3 **[Newman's lemma]** A rewriting system $S = \{(\ell_i, r_i)\}_{i \in I}$ is called length-reducing if $(\ell, r) \in S$ implies $|\ell| > |r|$. S is called confluent, if for all s, t, u with $s \to_S^* t$ and $s \to_S^* u$ there exists v with $t, u \to_S^* v$; it is local confluent, if s, t, u with $s \to_S t$ and $s \to_S u$ there exists v with $t, u \to_S^* v$.

Show that if S is length-reducing, then S is confluent if and only if it is local confluent.

Task 4 Show that each of the defined rewriting systems P_i is confluent and thus each term has a unique normal form (note that the rewriting system is length-reducing).

Task 5 An *involution* $\bar{\cdot}$ is any operation (defined in a semigroup) such that $\bar{\bar{\cdot}}$ is an identity and $\overline{ab} = \bar{b}\,\bar{a}$. In particular, we can define on some letters as an identity, such letters are called self-involuting.

Show that we can reduce a problem of word equations in a free semigroup with involution and regular constraints to the case in which there is no self-involuting letter.

Task 6 Show that if a homomorphism $\rho : M \mapsto \mathbb{B}_{n\times n}$ (so: Boolean matrices of size $n \times n$) from a free monoid with involution M into Boolean matrices does not preserve involution (in particular, the involution on $\mathbb{B}_{n\times n}$ may be undefined), then we can find a different set of Boolean matrices $\mathbb{B}_{m\times m}$ for which we define the involution and there is a homomorphism $\rho' : M \mapsto \mathbb{B}_{m\times m}$ from M to $\mathbb{B}_{m\times m}$ that preserves the involution and each set regular in $\mathbb{B}_{n\times n}$ is regular in $\mathbb{B}_{m\times m}$ (but not necessarily vice-versa).
Hint: [3]

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[1] First make the claim about the whole theory and then eliminate the negation as we did before.
[2] The simplest proof is through Kolmogorov's complexity, but random strings should also be good.
[3] Take $\overline{\mathbb{B}_{n\times n}}$ and consider $\mathbb{B}_{n\times n} \times \overline{\mathbb{B}_{n\times n}}$. How to define the involution?