Word Equations: Sheet 11

Task 1 Show that given a word equation over a free monoid with recognisable constraints given by ρ we can extend the alphabet Σ by

$$\{a_\tau \,:\, \tau \in N \text{ and there is a word } w \in \Sigma^* \text{ such that } \rho(w) = \tau\}.$$

Show the equisatisfiability of the problem over the original alphabet and over such an extended alphabet. Modify the algorithm that tests the satisfiability of word equations so that it works also in case of recognisable constraints. Can you implement the algorithm in PSPACE?

Below exercises present a different approach to generation of all solutions of a word equation.

Definition 1. A solution $S : \mathcal{X} \cup \Sigma \mapsto (\Sigma \cup \Sigma')^+$ of an equation $U = V$ over Σ is a *unifier* (with *free letters* Σ'), when $S(U)$ contains at least one letter from Σ'. S' is an *instance* of a unifier solution S, if for each variable X it holds that $S'(X) = \varphi(S(X))$ for some non-erasing non-permutating[1] morphism $\varphi : (\Sigma \cup \Sigma') \mapsto (\Sigma \cup \Sigma')^+$ that is constant on Σ. A solution S is *minimal*, if it is not a unifier solution, nor an instance of a unifier solution; it is a *minimal unifier* if it is a unifier solution and it is not an instance of another unifier solution.

Task 2 Suppose that S is a minimal solution of an equation $U = V$ and S' is a corresponding solution of $U' = V'$ obtained after popping or compression, i.e. $S(U) = S'(U')$. Show that if S is minimal then so is S'.

Task 3 Show that if we can construct a graph-like representation of all minimal and unifier-minimal solutions of a word equation then we can construct a graph-like representation of all solutions.

Task 4 Let S be a minimal solution of $U = V$. Show that:

- If ab is a substring of $S(U)$, where $a \neq b$, then ab is an explicit pair or a crossing pair.
- If a^k is a maximal block in $S(U)$ then a has an explicit occurrence in U or V and there is a visible occurrence of a^k.

Task 5 Using the above task show that if we can construct a graph-like representation of all minimal solutions of a word equation then we can construct a graph-like representation of all minimal and unifier-minimal solutions.

Task 6 Show that the term unification (so the case in which the variables can represent only well-formed terms) can be solved in linear time, in the sense that we can say, whether it has a solution or not. Some assumptions on the model may be needed.

[1] A morphism φ is *non-erasing* if $\varphi(a) \neq \epsilon$ for every letter a and it is *non-permutating* if φ is not a permutation on its domain.