Conjunctive grammars over a unary alphabet

Artur Jeż, Alexander Okhotin

September 7, 2007

Conjunctive and Boolean grammars

Context-free grammars: Rules of the form

$$A \to \alpha$$

"If w is generated by α, then w is generated by A".

Conjunctive and Boolean grammars

Context-free grammars: Rules of the form

$$A \to \alpha$$

"If w is generated by α, then w is generated by A".

✓ Multiple rules for A: *disjunction*.

Conjunctive and Boolean grammars

Context-free grammars: Rules of the form

$$A \to \alpha$$

"If w is generated by α, then w is generated by A".

✓ Multiple rules for A: *disjunction*.

Conjunctive grammars (Okhotin, 2000) Rules of the form

$$A \to \alpha_1 \& \ldots \& \alpha_m$$

"If w is generated by each α_i, then w is generated by A".

Conjunctive and Boolean grammars

Context-free grammars: Rules of the form

$$A \to \alpha$$

"If w is generated by α, then w is generated by A".

- ✓ Multiple rules for A: *disjunction*.

Conjunctive grammars (Okhotin, 2000) Rules of the form

$$A \to \alpha_1 \& \ldots \& \alpha_m$$

"If w is generated by each α_i, then w is generated by A".

Boolean grammars (Okhotin, 2003) Rules of the form

$$A \to \alpha_1 \& \ldots \& \alpha_m \& \neg\beta_1 \& \ldots \& \neg\beta_n$$

"If w is generated by each α_i and by none of β_j, then w is generated by A".

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \to \alpha_1 \& \ldots \& \alpha_m \quad \text{with } A \in N,\ \alpha_i \in (\Sigma \cup N)^*$$

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \to \alpha_1 \& \ldots \& \alpha_m \quad \text{with } A \in N,\ \alpha_i \in (\Sigma \cup N)^*$$

- Semantics by term rewriting:

$$\varphi(A) \Longrightarrow \varphi(\alpha_1 \& \ldots \& \alpha_m)$$

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \to \alpha_1 \& \ldots \& \alpha_m \quad \text{with } A \in N,\ \alpha_i \in (\Sigma \cup N)^*$$

- Semantics by term rewriting:

$$\varphi(A) \Longrightarrow \varphi(\alpha_1 \& \ldots \& \alpha_m)$$
$$\varphi(w \& \ldots \& w) \Longrightarrow \varphi(w)$$

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \to \alpha_1 \& \ldots \& \alpha_m \quad \text{with } A \in N,\ \alpha_i \in (\Sigma \cup N)^*$$

- Semantics by term rewriting:

$$\varphi(A) \Longrightarrow \varphi(\alpha_1 \& \ldots \& \alpha_m)$$
$$\varphi(w \& \ldots \& w) \Longrightarrow \varphi(w)$$

  - $L_G(A) = \{w \mid A \Longrightarrow^* w\}$

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \to \alpha_1 \& \ldots \& \alpha_m \quad \text{with } A \in N,\ \alpha_i \in (\Sigma \cup N)^*$$

- Semantics by term rewriting:

$$\varphi(A) \Longrightarrow \varphi(\alpha_1 \& \ldots \& \alpha_m)$$
$$\varphi(w \& \ldots \& w) \Longrightarrow \varphi(w)$$

  - $L_G(A) = \{w \mid A \Longrightarrow^* w\}$

- Semantics by language equations:

$$A = \bigcup_{A \to \alpha_1 \& \ldots \& \alpha_m \in P} \bigcap_{i=1}^{m} \alpha_i$$

Definition of conjunctive grammars

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \to \alpha_1 \& \ldots \& \alpha_m \quad \text{with } A \in N,\ \alpha_i \in (\Sigma \cup N)^*$$

- Semantics by term rewriting:

$$\varphi(A) \Longrightarrow \varphi(\alpha_1 \& \ldots \& \alpha_m)$$
$$\varphi(w \& \ldots \& w) \Longrightarrow \varphi(w)$$

  - $L_G(A) = \{w \mid A \Longrightarrow^* w\}$

- Semantics by language equations:

$$A = \bigcup_{A \to \alpha_1 \& \ldots \& \alpha_m \in P} \ \bigcap_{i=1}^{m} \alpha_i$$

  - $L_G(A)$ is the A-component of the least solution.

Properties of conjunctive and Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{wcw \mid w \in \{a,b\}^*\}$, etc., etc.

Example

$$S \to AE \& BC$$
$$A \to aA \mid \varepsilon$$
$$B \to aBb \mid \varepsilon$$
$$C \to cC \mid \varepsilon$$
$$E \to bEc \mid \varepsilon$$

Properties of conjunctive and Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{wcw \mid w \in \{a, b\}^*\}$, etc., etc.

Example

$$
\begin{aligned}
S &\to AE \& BC & L_S &= (L_A \cdot L_E) \cap (L_B \cdot L_C) \\
A &\to aA \mid \varepsilon & L_A &= \{a\} \cdot L_A \cup \{\varepsilon\} \\
B &\to aBb \mid \varepsilon & L_B &= \{a\} \cdot L_B \cdot \{b\} \cup \{\varepsilon\} \\
C &\to cC \mid \varepsilon & L_C &= \{c\} \cdot L_C \cup \{\varepsilon\} \\
E &\to bEc \mid \varepsilon & L_E &= \{b\} \cdot L_E \cdot \{c\} \cup \{\varepsilon\}
\end{aligned}
$$

Properties of conjunctive and Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{wcw \mid w \in \{a,b\}^*\}$, etc., etc.

Example

$S \to AE \& BC$	$L_S = (L_A \cdot L_E) \cap (L_B \cdot L_C)$	$\{a^n b^n c^n : n \in \mathbb{N}\}$
$A \to aA \mid \varepsilon$	$L_A = \{a\} \cdot L_A \cup \{\varepsilon\}$	a^*
$B \to aBb \mid \varepsilon$	$L_B = \{a\} \cdot L_B \cdot \{b\} \cup \{\varepsilon\}$	$\{a^n b^n : n \in \mathbb{N}\}$
$C \to cC \mid \varepsilon$	$L_C = \{c\} \cdot L_C \cup \{\varepsilon\}$	c^*
$E \to bEc \mid \varepsilon$	$L_E = \{b\} \cdot L_E \cdot \{c\} \cup \{\varepsilon\}$	$\{b^n c^n : n \in \mathbb{N}\}$

Properties of conjunctive and Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{wcw \mid w \in \{a,b\}^*\}$, etc., etc.

Example

$S \to AE\&BC$	$L_S = (L_A \cdot L_E) \cap (L_B \cdot L_C)$	$\{a^n b^n c^n : n \in \mathbb{N}\}$
$A \to aA \mid \varepsilon$	$L_A = \{a\} \cdot L_A \cup \{\varepsilon\}$	a^*
$B \to aBb \mid \varepsilon$	$L_B = \{a\} \cdot L_B \cdot \{b\} \cup \{\varepsilon\}$	$\{a^n b^n : n \in \mathbb{N}\}$
$C \to cC \mid \varepsilon$	$L_C = \{c\} \cdot L_C \cup \{\varepsilon\}$	c^*
$E \to bEc \mid \varepsilon$	$L_E = \{b\} \cdot L_E \cdot \{c\} \cup \{\varepsilon\}$	$\{b^n c^n : n \in \mathbb{N}\}$

- Languages contained in $DTIME(n^3) \cap DSPACE(n)$.

Properties of conjunctive and Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{wcw \mid w \in \{a,b\}^*\}$, etc., etc.

Example

$S \to AE\&BC$	$L_S = (L_A \cdot L_E) \cap (L_B \cdot L_C)$	$\{a^n b^n c^n : n \in \mathbb{N}\}$
$A \to aA \mid \varepsilon$	$L_A = \{a\} \cdot L_A \cup \{\varepsilon\}$	a^*
$B \to aBb \mid \varepsilon$	$L_B = \{a\} \cdot L_B \cdot \{b\} \cup \{\varepsilon\}$	$\{a^n b^n : n \in \mathbb{N}\}$
$C \to cC \mid \varepsilon$	$L_C = \{c\} \cdot L_C \cup \{\varepsilon\}$	c^*
$E \to bEc \mid \varepsilon$	$L_E = \{b\} \cdot L_E \cdot \{c\} \cup \{\varepsilon\}$	$\{b^n c^n : n \in \mathbb{N}\}$

- Languages contained in $DTIME(n^3) \cap DSPACE(n)$.
- Linear case: equivalent to one-way real-time CA.

Properties of conjunctive and Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{wcw \mid w \in \{a,b\}^*\}$, etc., etc.

Example

$S \to AE \& BC$	$L_S = (L_A \cdot L_E) \cap (L_B \cdot L_C)$	$\{a^n b^n c^n : n \in \mathbb{N}\}$
$A \to aA \mid \varepsilon$	$L_A = \{a\} \cdot L_A \cup \{\varepsilon\}$	a^*
$B \to aBb \mid \varepsilon$	$L_B = \{a\} \cdot L_B \cdot \{b\} \cup \{\varepsilon\}$	$\{a^n b^n : n \in \mathbb{N}\}$
$C \to cC \mid \varepsilon$	$L_C = \{c\} \cdot L_C \cup \{\varepsilon\}$	c^*
$E \to bEc \mid \varepsilon$	$L_E = \{b\} \cdot L_E \cdot \{c\} \cup \{\varepsilon\}$	$\{b^n c^n : n \in \mathbb{N}\}$

- Languages contained in $DTIME(n^3) \cap DSPACE(n)$.
- Linear case: equivalent to one-way real-time CA.
- Practical parsing methods: recursive descent, generalized LR.

The case of a unary alphabet

- $\Sigma = \{a\}$.

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n
- Language $\longleftrightarrow$ set of numbers

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n
- Language $\longleftrightarrow$ set of numbers
- $K \cdot L$ $\longleftrightarrow$ $X \boxplus Y = \{x + y \mid x \in X, y \in Y\}$

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n
- Language $\longleftrightarrow$ set of numbers
- $K \cdot L$ $\longleftrightarrow$ $X \boxplus Y = \{x + y \mid x \in X, y \in Y\}$
- Regular $\longleftrightarrow$ ultimately periodic

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n
- Language $\longleftrightarrow$ set of numbers
- $K \cdot L$ $\longleftrightarrow$ $X \boxplus Y = \{x + y \mid x \in X, y \in Y\}$
- Regular $\longleftrightarrow$ ultimately periodic

Theorem (Bar-Hillel et al., 1961)

Every context-free language over $\{a\}$ is regular.

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n
- Language $\longleftrightarrow$ set of numbers
- $K \cdot L$ $\longleftrightarrow$ $X \boxplus Y = \{x + y \mid x \in X, y \in Y\}$
- Regular $\longleftrightarrow$ ultimately periodic

Theorem (Bar-Hillel et al., 1961)

Every context-free language over $\{a\}$ is regular.

Problem

The power of conjunctive grammars over $\{a\}$?

The case of a unary alphabet

- $\Sigma = \{a\}$.
- a^n $\longleftrightarrow$ number n
- Language $\longleftrightarrow$ set of numbers
- $K \cdot L$ $\longleftrightarrow$ $X \boxplus Y = \{x + y \mid x \in X, y \in Y\}$
- Regular $\longleftrightarrow$ ultimately periodic

Theorem (Bar-Hillel et al., 1961)

Every context-free language over $\{a\}$ is regular.

Problem

The power of conjunctive grammars over $\{a\}$?

- Can generate $\{a^{4^n} \mid n \geqslant 0\}$ (Jeż, 2007).

Using positional notation

- Our approach: using base-k notation.

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n
- $\Sigma_k = \{0, 1, \ldots, k-1\}$, strings in $\Sigma_k^* \setminus 0\Sigma_k^*$.

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n
- $\Sigma_k = \{0, 1, \ldots, k-1\}$, strings in $\Sigma_k^* \setminus 0\Sigma_k^*$.
- Isomorphism $\Sigma_k^* \setminus 0\Sigma_k^* \leftrightarrow a^*$.

$$f_k(k\text{-ary notation of } n) = a^n$$

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n
- $\Sigma_k = \{0, 1, \ldots, k-1\}$, strings in $\Sigma_k^* \setminus 0\Sigma_k^*$.
- Isomorphism $\Sigma_k^* \setminus 0\Sigma_k^* \leftrightarrow a^*$.

$$f_k(k\text{-ary notation of } n) = a^n$$

- Extends to languages:

$$f_k(L) = \{f_k(w) \mid w \in L\}$$

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n
- $\Sigma_k = \{0, 1, \ldots, k-1\}$, strings in $\Sigma_k^* \setminus 0\Sigma_k^*$.
- Isomorphism $\Sigma_k^* \setminus 0\Sigma_k^* \leftrightarrow a^*$.

$$f_k(k\text{-ary notation of } n) = a^n$$

- Extends to languages:

$$f_k(L) = \{f_k(w) \mid w \in L\}$$

Example

$$f_4(10^*) = \{a^{4^n} \mid n \geqslant 0\}$$

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n
- $\Sigma_k = \{0, 1, \ldots, k-1\}$, strings in $\Sigma_k^* \setminus 0\Sigma_k^*$.
- Isomorphism $\Sigma_k^* \setminus 0\Sigma_k^* \leftrightarrow a^*$.

$$f_k(k\text{-ary notation of } n) = a^n$$

- Extends to languages:

$$f_k(L) = \{f_k(w) \mid w \in L\}$$

Example

$$f_4(10^*) = \{a^{4^n} \mid n \geqslant 0\}$$

- Equations over Σ_k^* with $\cap, \cup, \boxplus$

Using positional notation

- Our approach: using base-k notation.
- $a^n \quad \longleftrightarrow \quad k$-ary notation of n
- $\Sigma_k = \{0, 1, \dots, k-1\}$, strings in $\Sigma_k^* \setminus 0\Sigma_k^*$.
- Isomorphism $\Sigma_k^* \setminus 0\Sigma_k^* \leftrightarrow a^*$.

$$f_k(k\text{-ary notation of } n) = a^n$$

- Extends to languages:

$$f_k(L) = \{f_k(w) \mid w \in L\}$$

Example

$$f_4(10^*) = \{a^{4^n} \mid n \geqslant 0\}$$

- Equations over Σ_k^* with $\cap, \cup, \boxplus$
- Isomorphism between language equations.

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	$Y_1 = (Y_2 Y_2 \cap Y_1 Y_3) \cup a$
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	$Y_2 = (Y_{12} Y_2 \cap Y_1 Y_1) \cup aa$
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	$Y_3 = (Y_{12} Y_{12} \cap Y_1 Y_2) \cup aaa$
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	$Y_{12} = (Y_3 Y_3 \cap Y_1 Y_2)$

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	$Y_1 = (Y_2 Y_2 \cap Y_1 Y_3) \cup a$
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	$Y_2 = (Y_{12} Y_2 \cap Y_1 Y_1) \cup aa$
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	$Y_3 = (Y_{12} Y_{12} \cap Y_1 Y_2) \cup aaa$
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	$Y_{12} = (Y_3 Y_3 \cap Y_1 Y_2)$
base 4: $(10^*, 20^*, 30^*, 120^*)$	

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	$Y_1 = (Y_2 Y_2 \cap Y_1 Y_3) \cup a$
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	$Y_2 = (Y_{12} Y_2 \cap Y_1 Y_1) \cup aa$
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	$Y_3 = (Y_{12} Y_{12} \cap Y_1 Y_2) \cup aaa$
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	$Y_{12} = (Y_3 Y_3 \cap Y_1 Y_2)$
base 4: $(10^*, 20^*, 30^*, 120^*)$	$(\{a^{4^n}\}, \{a^{2 \cdot 4^n}\}, \{a^{3 \cdot 4^n}\}, \{a^{6 \cdot 4^n}\})$

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	$Y_1 = (Y_2 Y_2 \cap Y_1 Y_3) \cup a$
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	$Y_2 = (Y_{12} Y_2 \cap Y_1 Y_1) \cup aa$
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	$Y_3 = (Y_{12} Y_{12} \cap Y_1 Y_2) \cup aaa$
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	$Y_{12} = (Y_3 Y_3 \cap Y_1 Y_2)$
base 4: $(10^*, 20^*, 30^*, 120^*)$	$(\{a^{4^n}\}, \{a^{2\cdot 4^n}\}, \{a^{3\cdot 4^n}\}, \{a^{6\cdot 4^n}\})$

- $X_2 \boxplus X_2 = 20^* \boxplus 20^* = 10^+ \cup 20^* 20^*$

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	$Y_1 = (Y_2 Y_2 \cap Y_1 Y_3) \cup a$
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	$Y_2 = (Y_{12} Y_2 \cap Y_1 Y_1) \cup aa$
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	$Y_3 = (Y_{12} Y_{12} \cap Y_1 Y_2) \cup aaa$
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	$Y_{12} = (Y_3 Y_3 \cap Y_1 Y_2)$
base 4: $(10^*, 20^*, 30^*, 120^*)$	$(\{a^{4^n}\}, \{a^{2\cdot 4^n}\}, \{a^{3\cdot 4^n}\}, \{a^{6\cdot 4^n}\})$

- $X_2 \boxplus X_2 = 20^* \boxplus 20^* = 10^+ \cup 20^*20^*$
- $X_1 \boxplus X_3 = 10^* \boxplus 30^* = 10^+ \cup 10^*30^* \cup 30^*10^*$,

Nonperiodic unary conjunctive languages

Example (Jeż, DLT 2007)

$\mathbb{N}$	a^*
$X_1 = (X_2 \boxplus X_2 \cap X_1 \boxplus X_3) \cup \{1\}$	$Y_1 = (Y_2 Y_2 \cap Y_1 Y_3) \cup a$
$X_2 = (X_{12} \boxplus X_2 \cap X_1 \boxplus X_1) \cup \{2\}$	$Y_2 = (Y_{12} Y_2 \cap Y_1 Y_1) \cup aa$
$X_3 = (X_{12} \boxplus X_{12} \cap X_1 \boxplus X_2) \cup \{3\}$	$Y_3 = (Y_{12} Y_{12} \cap Y_1 Y_2) \cup aaa$
$X_{12} = X_3 \boxplus X_3 \cap X_1 \boxplus X_2$	$Y_{12} = (Y_3 Y_3 \cap Y_1 Y_2)$
base 4: $(10^*, 20^*, 30^*, 120^*)$	$(\{a^{4^n}\}, \{a^{2 \cdot 4^n}\}, \{a^{3 \cdot 4^n}\}, \{a^{6 \cdot 4^n}\})$

- $X_2 \boxplus X_2 = 20^* \boxplus 20^* = 10^+ \cup 20^* 20^*$
- $X_1 \boxplus X_3 = 10^* \boxplus 30^* = 10^+ \cup 10^* 30^* \cup 30^* 10^*$,
- $(X_2 \boxplus X_2) \cap (X_1 \boxplus X_3) = 10^+$.

More unary conjunctive languages

Theorem (Jeż, DLT 2007)

For any regular language $R \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a conjunctive grammar for $f_k(R)$.

More unary conjunctive languages

Theorem (Jeż, DLT 2007)

For any regular language $R \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a conjunctive grammar for $f_k(R)$.

Note

$f_k(R)$ has linear or exponential growth.

More unary conjunctive languages

Theorem (Jeż, DLT 2007)

For any regular language $R \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a conjunctive grammar for $f_k(R)$.

Note

$f_k(R)$ has linear or exponential growth.

* * *

Theorem

For every trellis automaton M over Σ_k with $L(M) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a conjunctive grammar for $f_k(L(M))$.

Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a
$M = (\Sigma, Q, I, \delta, F)$ where:

Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;

Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a
$M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;

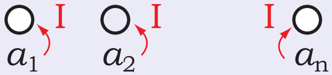
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;



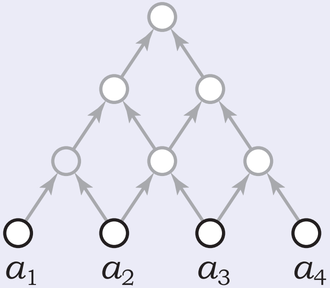
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;



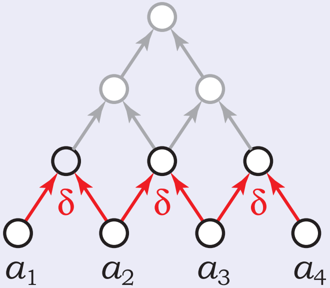
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;



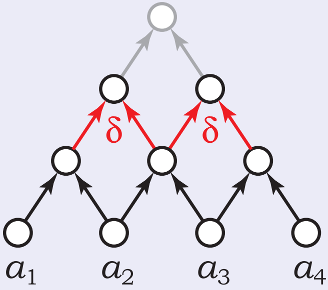
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;



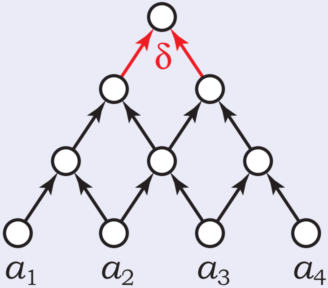
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;



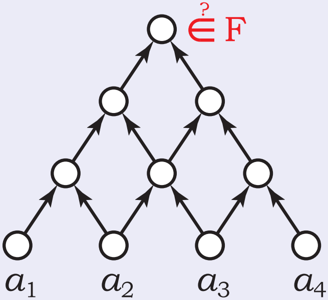
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;
- $F \subset Q$: final states.



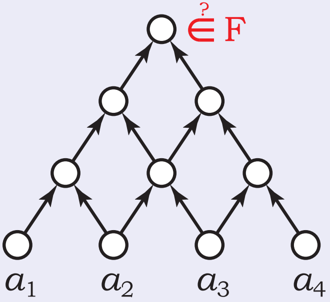
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;
- $F \subset Q$: final states.



- Equivalent to linear conjunctive grammars.

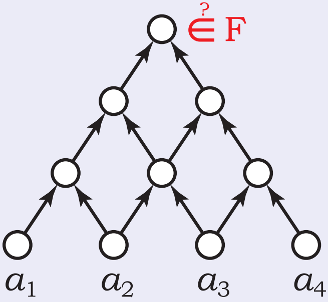
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;
- $F \subset Q$: final states.



- Equivalent to linear conjunctive grammars.
- Closed under $\cup, \cap, \sim$, not closed under concatenation.

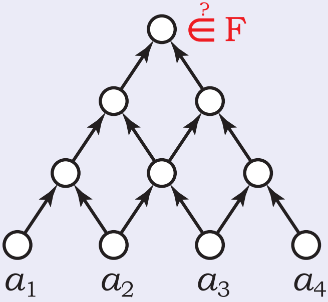
Trellis automata

(one-way real-time cellular automata)

Definition

A trellis automaton is a $M = (\Sigma, Q, I, \delta, F)$ where:

- Σ: input alphabet;
- Q: finite set of states;
- $I : \Sigma \to Q$ sets initial states;
- $\delta : Q \times Q \to Q$, transition function;
- $F \subset Q$: final states.



- Equivalent to linear conjunctive grammars.
- Closed under $\cup, \cap, \sim$, not closed under concatenation.
- Can recognize $\{wcw\}$, $\{a^n b^n c^n\}$, $\{a^n b^{2^n}\}$, VALC.

Main lemma

Lemma

For every trellis automaton M over Σ_k with $L(M) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a system with $\cup$, $\cap$, $\boxplus$ and regular constants, with least solution

$$\{1w10^* \mid w \boxplus 1 \in L(M)\}, \ldots,$$

Main lemma

Lemma

For every trellis automaton M over Σ_k with $L(M) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a system with $\cup$, $\cap$, $\boxplus$ and regular constants, with least solution

$$\{1w10^* \mid w \boxplus 1 \in L(M)\}, \ldots,$$

- $1w10^*$ represents w.

Main lemma

Lemma

For every trellis automaton M over Σ_k with $L(M) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, there exists a system with $\cup$, $\cap$, $\boxplus$ and regular constants, with least solution

$$\{1w10^* \mid w \boxplus 1 \in L(M)\}, \dots ,$$

- $1w10^*$ represents w.
- Regular constants, can be changed to singleton.

The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.

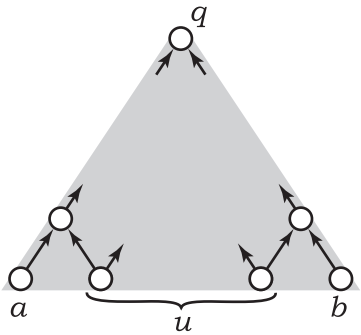
The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

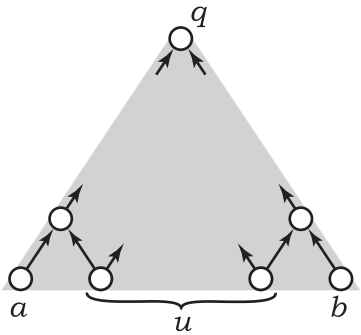
- $aub \in L_M(q)$



The construction

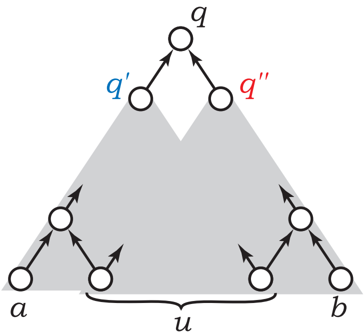
- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

- $aub \in L_M(q) \Leftrightarrow$



The construction

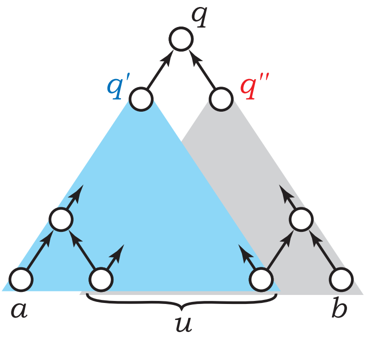
- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$
- $aub \in L_M(q) \Leftrightarrow$
  $\exists q', q'' : \delta(q', q'') = q,$



The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

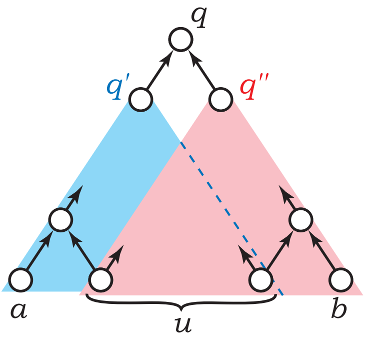
- $aub \in L_M(q) \Leftrightarrow$
  $\exists q', q'' : \delta(q', q'') = q$,
  $au \in L_M(q')$,



The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

- $aub \in L_M(q) \Leftrightarrow$
  $\exists q', q'' : \delta(q', q'') = q$,
  $au \in L_M(q')$,
  $ub \in L_M(q'')$.

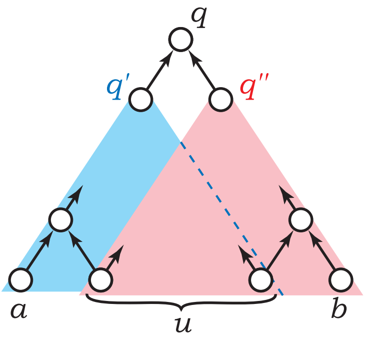


The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

- $aub \in L_M(q) \Leftrightarrow$
  $\exists q', q'' : \delta(q', q'') = q$,
  $au \in L_M(q')$,
  $ub \in L_M(q'')$.
- Let $1au10^* \subseteq X_{q'}$, $1ub10^* \subseteq X_{q''}$.

$$X_q = \bigcup_{\substack{q', q'' : \delta(q', q'') = q \\ a, b \in \Sigma_k}} \rho_b(X_{q'}) \cap \lambda_a(X_{q''})$$

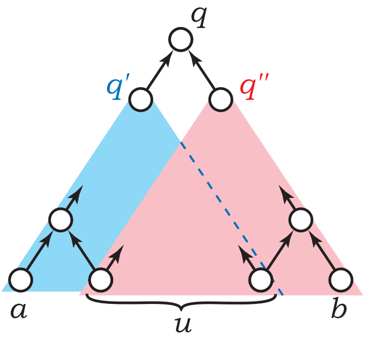


The construction

- Set of variables $\{X_q \mid q \in Q\}$, representing $\{L_M(q) \mid q \in Q\}$.
- Actually, $X_q = \{1w10^* \mid w \boxplus 1 \in L_M(q)\}$

- $aub \in L_M(q) \Leftrightarrow$
  $\exists q', q'' : \delta(q', q'') = q$,
  $au \in L_M(q')$,
  $ub \in L_M(q'')$.
- Let $1au10^* \subseteq X_{q'}$, $1ub10^* \subseteq X_{q''}$.

$$X_q = \bigcup_{\substack{q',q'':\delta(q',q'')=q \\ a,b\in\Sigma_k}} \rho_b(X_{q'}) \cap \lambda_a(X_{q''})$$



$$\lambda_a(1w10^k) = 1aw10^k$$

$$\rho_b(1w10^k) = 1wb10^{k-1}$$

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word
$1w10^\ell$		

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word
$1w10^\ell = 1w'j'10^\ell$		

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word
$1w10^\ell = 1w'j'10^\ell$	$\cap 1\Sigma_k^* j' 10^*$	$1w'j'10^\ell$

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \Bigg(\Big((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \Big) \boxplus (j-2)10^* \Bigg) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word
$1w10^\ell = 1w'j'10^\ell$	$\cap 1\Sigma_k^* j' 10^*$	$1w'j'10^\ell$
$1w'j'10^\ell$	$\boxplus 10^* \cap 1\Sigma_k^* j' 20^*$	$1w'j'20^\ell$

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word
$1w10^\ell = 1w'j'10^\ell$	$\cap 1\Sigma_k^* j'10^*$	$1w'j'10^\ell$
$1w'j'10^\ell$	$\boxplus 10^* \cap 1\Sigma_k^* j'20^*$	$1w'j'20^\ell$
$1w'j'20^\ell$	$\boxplus (j-2)10^* \cap 1\Sigma_k^* j10^*$	$1w'j'j10^{\ell-1}$

How does ρ look like

The equations for ρ_j:

$$\rho_j(X) = \bigcup_{j'} \left(\left((X \cap 1\Sigma_k^* j' 10^* \boxplus 10^*) \cap 1\Sigma_k^* j' 20^* \right) \boxplus (j-2)10^* \right) \cap 1\Sigma_k^* j 10^*$$

Word	Operation	New word
$1w10^\ell = 1w'j'10^\ell$	$\cap 1\Sigma_k^* j'10^*$	$1w'j'10^\ell$
$1w'j'10^\ell$	$\boxplus 10^* \cap 1\Sigma_k^* j'20^*$	$1w'j'20^\ell$
$1w'j'20^\ell$	$\boxplus (j-2)10^* \cap 1\Sigma_k^* j10^*$	$1w'j'j10^{\ell-1}$
$1w'j'j10^{\ell-1}$	$\bigcup_{j'}$	$1wj10^{\ell-1}$

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$, determine whether $L(G) = \varnothing$"

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$,
determine whether $L(G) = \varnothing$" — *undecidable.*

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$*,*
determine whether $L(G) = \varnothing$*"* — *undecidable.*

- Turing machine T recognizes X;

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$, determine whether $L(G) = \varnothing$" — undecidable.

- Turing machine T recognizes X;
- Trellis automaton M for $\text{VALC}(T)$;

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$, determine whether $L(G) = \varnothing$" — *undecidable*.

- Turing machine T recognizes X;
- Trellis automaton M for VALC(T);
- Conjunctive grammar G for $f_k(\text{VALC}(T))$;

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$, determine whether $L(G) = \varnothing$" — *undecidable*.

- Turing machine T recognizes X;
- Trellis automaton M for $\mathrm{VALC}(T)$;
- Conjunctive grammar G for $f_k(\mathrm{VALC}(T))$;
- $L(G) = \varnothing \Leftrightarrow L(T) = \varnothing$

Undecidable properties

Proposition

"Given conjunctive grammar G over $\{a\}$, determine whether $L(G) = \varnothing$" — undecidable.

- Turing machine T recognizes X;
- Trellis automaton M for $\mathrm{VALC}(T)$;
- Conjunctive grammar G for $f_k(\mathrm{VALC}(T))$;
- $L(G) = \varnothing \Leftrightarrow L(T) = \varnothing$

Theorem

For every fixed conjunctive $L_0 \subseteq a^*$, the problem "Given conjunctive grammar G over $\{a\}$, determine whether $L(G) = L_0$" — undecidable.

Unbounded growth

Theorem

$\forall$ *r. e. set* $X = \{g_X(n) \mid n \geqslant 1\}$, *with* $g_X \nearrow$
$\exists$ *conjunctive grammar* G *with* $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

Unbounded growth

Theorem

$\forall$ r. e. set $X = \{g_X(n) \mid n \geqslant 1\}$, with $g_X \nearrow$
$\exists$ conjunctive grammar G with $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

$$g_G(n) > g_X(n) \quad (\forall n \geqslant 1)$$

Unbounded growth

Theorem

$\forall$ r. e. set $X = \{g_X(n) \mid n \geqslant 1\}$, with $g_X \nearrow$
$\exists$ conjunctive grammar G with $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

$$g_G(n) > g_X(n) \quad (\forall n \geqslant 1)$$

- Turing machine T recognizes X;

Unbounded growth

Theorem

$\forall$ r. e. set $X = \{g_X(n) \mid n \geqslant 1\}$, with $g_X \nearrow$
$\exists$ conjunctive grammar G with $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

$$g_G(n) > g_X(n) \quad (\forall n \geqslant 1)$$

- Turing machine T recognizes X;
- Trellis automaton M for VALC(T);

Unbounded growth

Theorem

$\forall$ r. e. set $X = \{g_X(n) \mid n \geqslant 1\}$, with $g_X \nearrow$
$\exists$ conjunctive grammar G with $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

$$g_G(n) > g_X(n) \quad (\forall n \geqslant 1)$$

- Turing machine T recognizes X;
- Trellis automaton M for VALC(T);
- Conjunctive grammar G for f_k(VALC(T));

Unbounded growth

Theorem

$\forall$ r. e. set $X = \{g_X(n) \mid n \geqslant 1\}$, with $g_X \nearrow$
$\exists$ conjunctive grammar G with $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

$$g_G(n) > g_X(n) \quad (\forall n \geqslant 1)$$

- Turing machine T recognizes X;
- Trellis automaton M for VALC(T);
- Conjunctive grammar G for $f_k(\text{VALC}(T))$;
- $g_G(n) > g_X(n)$.

Unbounded growth

Theorem

$\forall$ r. e. set $X = \{g_X(n) \mid n \geqslant 1\}$, with $g_X \nearrow$
$\exists$ conjunctive grammar G with $L(G) = \{a^{g_G(n)} \mid n \geqslant 1\}$:

$$g_G(n) > g_X(n) \quad (\forall n \geqslant 1)$$

- Turing machine T recognizes X;
- Trellis automaton M for VALC(T);
- Conjunctive grammar G for $f_k(\text{VALC}(T))$;
- $g_G(n) > g_X(n)$.

Remark

Polynomial growth can be achieved.

Conclusion

- Conjunctive grammar: CFG with intersection.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of VALC(T) is conjunctive.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of VALC(T) is conjunctive.
  - Undecidability.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of VALC(T) is conjunctive.
  - Undecidability.
  - Growth not recursively bounded.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of $\mathrm{VALC}(T)$ is conjunctive.
  - Undecidability.
  - Growth not recursively bounded.
- Research problems.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of $\mathrm{VALC}(T)$ is conjunctive.
  - Undecidability.
  - Growth not recursively bounded.
- Research problems.
  - Closure under complementation.

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of VALC(T) is conjunctive.
  - Undecidability.
  - Growth not recursively bounded.
- Research problems.
  - Closure under complementation.
  - Separation from DSPACE(n).

Conclusion

- Conjunctive grammar: CFG with intersection.
- Specifying unary notation of nontrivial languages.
  - $\{a^{f_k(w)} \mid w \in L(M)\}$ for any TA M.
  - Equations over sets of integers.
  - Positional notation.
- Unary notation of VALC(T) is conjunctive.
  - Undecidability.
  - Growth not recursively bounded.
- Research problems.
  - Closure under complementation.
  - Separation from DSPACE(n).
  - Automaton representation.