DFA hyper-minimisation

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DFA minimisation

Definition

DFA: $\langle Q, \Sigma, \delta, q_0, F \rangle$, where $\delta : Q \times \Sigma \mapsto Q$. DFA is minimal, if it has the minimal number of states among automata recognising $L(M)$.

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- unique with this property
- calculated using $\equiv_L$:

$$w \equiv w' \text{ if and only if } \forall w'' \;\; ww'' \in L \iff w'w'' \in L$$

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  - partition of states of M
- Hopcroft's algorithm: $\mathcal{O}(n \log n)$; refines the partition of states

f-equivalence and hyper-minimisation

Definition (f-equivalent)

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Remark

For fixed L we extend $\sim$ to words: $w \sim w' \iff w^{-1}L \sim w'^{-1}L$
For fixed automata M we extend $\sim$ to states: $q \sim q' \iff L(q) \sim L(q')$
(where $L(q)$ is the language recognised starting from q).

Approach

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We want a relation on words, such that equivalence classes are states of a hyper-minimal automaton, $\sim$ *is a natural candidate.*

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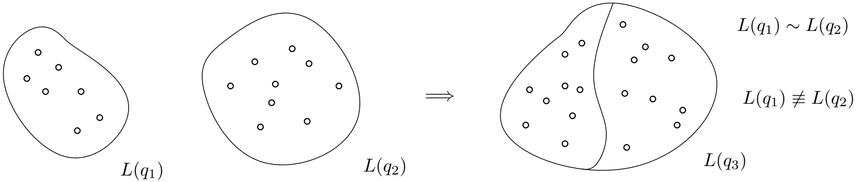
- Classes of $\sim$ are groups of classes of $\equiv$.
- We cannot greedily merge those groups: $w : \delta(q_0, w) = q_1$: $wL(q_1)$ changes to $wL(q_3) \neq wL(q_1)$. Infinitely many such w — problem!



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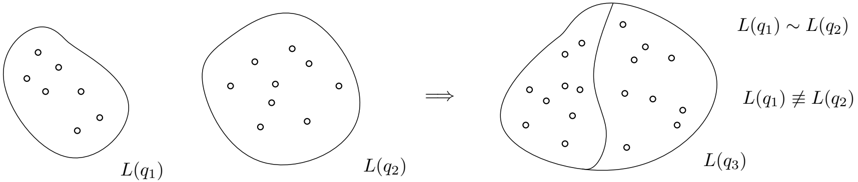

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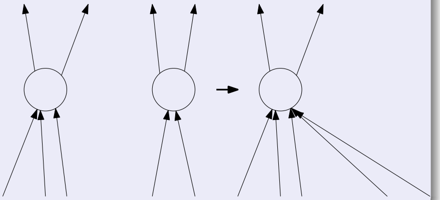
Definition

State q is in preamble if $\{w : \delta(q_0, w) = q\}$ is finite. In kernel otherwise.

Heuristic

Definition (state merging)



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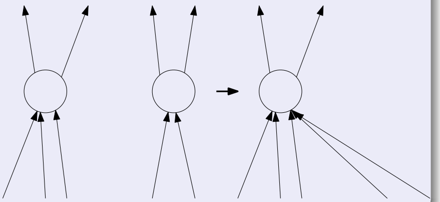

Heuristic

Greedily merge q to p whenever

- $q \equiv p$ *or*
- $q \sim p$ *and q is in the preamble*

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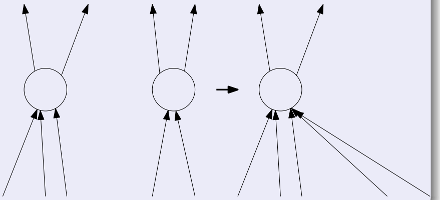

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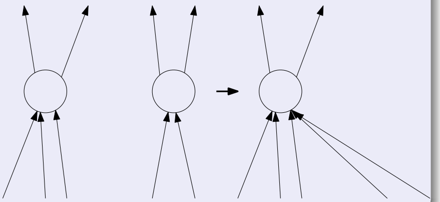

Heuristic

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- $q \sim p$ *and q is in the preamble and there is no path from p to q*

Theorem (A. Badr, V. Geffert, I. Shipman)

The heuristic is proper, i.e. it results in hyper-minimal automaton f-equivalent to the input one.

Data structures

Definition (Operational definition of $\sim$)

- $D^M(q, q')$ if $q = q'$ or,
- $D^M(q, q')$ if for all $a \in \Sigma$ $D^M\big(\delta_M(q, a), \delta_M(q', a)\big)$.

Lemma

If the automaton M is minimised the D coincides with $\sim$.

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We need a dictionary structure supporting

- query, if there are q, q' such that $(\delta(q, 0), \delta(q, 1)) = (\delta(q', 0), \delta(q', 1))$
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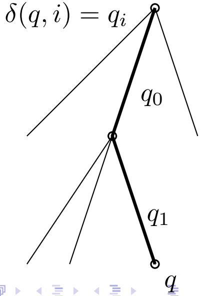
- Deterministic — tree: the path from root to the leave is $(\delta(q, 0), \delta(q, 1))$
- Randomised — hashing



Algorithm

Calculating relation D over states

- identify q, q' with the same successors
- delete the one with less predecessors
- update the predecessors

Using D greedily merge states.

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Using D greedily merge states.
Running time: $\mathcal{O}(n \log n)$ times insertion time

- insertion time:
  - deterministic: $\mathcal{O}(\log n)$
  - randomised $\mathcal{O}(1)$

Remarks and Questions

- $|\Sigma|$ has linear impact on the running time
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- Deterministic running time $\mathcal{O}(n \log n)$?
- Checking the f-equivalence of two automata is faster?

Refinment

Definition (distance between languages)

$$d(L, L') = \begin{cases} \max\{|u| : u \in L(w) \Delta L(w')\} + 1 & \text{if } L \neq L' \ , \\ 0 & \text{if } L = L' \ . \end{cases}$$

Definition (k-f-equivalence)

$L \sim_k L' \iff d(L, L') \leq k$

Definition

M is k-minimal if it has the least number of states among the $\sim_k$ automata.

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Algorithm is similar, but some theoretical work is to be done.

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- *Suppose there are* w_1, w_2 *with respective* q_1, q_2 *and* $L(w_1), L(w_2)$*.*
- *We merge state* q_1 *to* q_2
- *Intuitively,* $w_1 L(w_1)$ *changes to* $w_1 L(w_2)$
- *If* $L(w_1) \neq L(w_2)$ *we want*
  $k \geq d(w_1 L(w_1); w_1 L(w_2)) = |w_1| + d(L(w_1), L(w_2))$

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$w_1 \sim_k w_2 \iff L(w_1) = L(w_2)$ or $\min(|w_1|, |w_2|) + d(L(w_1), L(w_2)) \leq k$

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This is not an equivalence relation: it is not transitive.

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Lemma

If $\{w_i\}_{i=1}^{\ell}$ *satisfy* $w_i \not\sim_k w_j$ *then every automaton* k*-*f*-equivalent to* M *has at least* ℓ *states.*

Adjusting the relation

Definition (Expanding for states)

For q define its representative word $\text{word}(w)$: the longest word w such that $\delta(q_0, w) = q$. (take any word of length $k+1$ if this is badly defined).
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Improving $\sim_k$ to an equivalence relation $\approx_k$ satisfying:

- $w \approx_k w'$ implies $w \sim_k w'$
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- $w \not\approx_k w'$ implies $\text{Rep}(w) \not\sim_k \text{Rep}(w')$

Lemma

$\approx_k$ can be calculated out of $\sim_k$ in a greedy fashion (using word*)*

k-minimal Automata

Definition (k-minimal automata N)

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Lemma

$N \sim_k M$

Proof.

- for $\text{Rep}(q)$ s.t. $|\text{Rep}(q)| > k$ transition structure does not change.
- for other states by backward induction we show that $d(L_M(q), L_N(\text{Rep}(q))) \leq k$ □

It is k-minimal by previous lemma.

Remark

Algorithm — refinement of the previous one

Questions

- Deterministic running time $\mathcal{O}(n \log n)$?
- Checking the k-f-equivalence of two automata is faster?