Sorting Signed Permutations by Reversals in Nearly-Linear Time

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Input: Signed permuation π on *n* elements
Output: Shortest sequence of reversals sorting π

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$$1 \quad 2 \quad -5 \quad -4 \quad -3 \quad 6$$

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$$
\begin{array}{cccccc}
4 & 5 & -2 & -6 & 3 & 1 \\
4 & 5 & -2 & -1 & -3 & 6 \\
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\end{array}
$$

Transforming cabbage into turnip

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Brassica oleracea (cabbage)

Brassica campestris (turnip)

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$\overrightarrow{1}\ \overleftarrow{-5}\ \overrightarrow{4}\ \overleftarrow{-3}\ \overrightarrow{2}$

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Finding the shortest sequence of reversals

Early 90s: upper and lower bounds, approximation

Year	Authors	Runtime
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2024	this work	$\mathcal{O}(n \log^2 n / \log \log n)$

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Some insights from Hannenhalli-Pevzner

The goal is to perform oriented reversals that create a new *adjacency*: $i, i+1$ or $-(i+1), -i$, e.g.:

5 -6 2 -4 3 -1 7 $\rightarrow$ 5 1 -3 4 -2 **6** **7**

5 -6 2 -4 3 -1 7 $\rightarrow$ -2 6 **-5** **-4** 3 1 7

Sometimes such a reversal does not exist, but we can do an $\mathcal{O}(n)$-time preprocessing and later focus only on finding the oriented reversals [Kaplan, Shamir, Tarjan 1999].

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Operations on a black-white graph

Input:	Graph with nodes colored black or white
Output:	The shortest sequence of toggle operations that transform it to an independent set of white nodes

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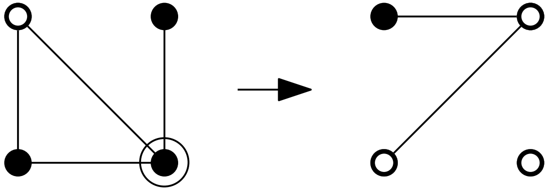

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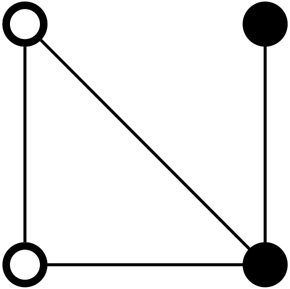

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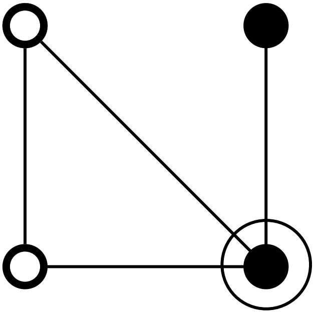

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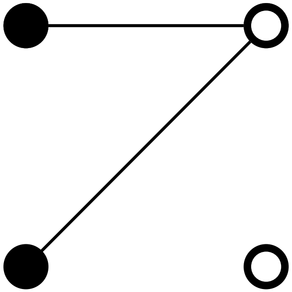

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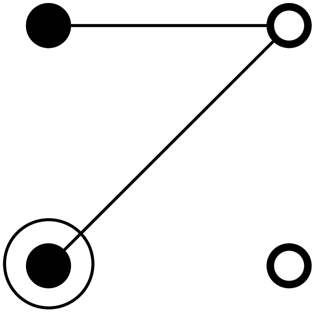

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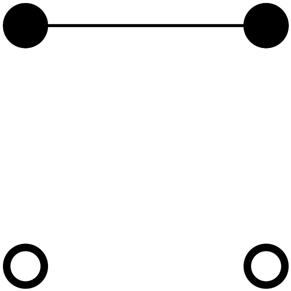

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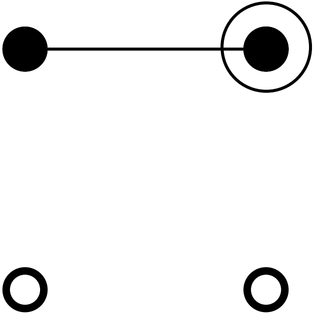

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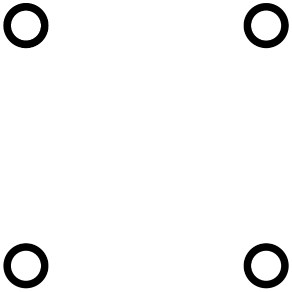

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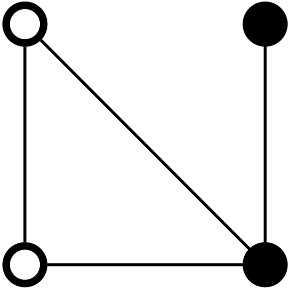

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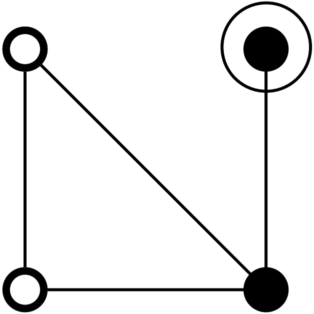

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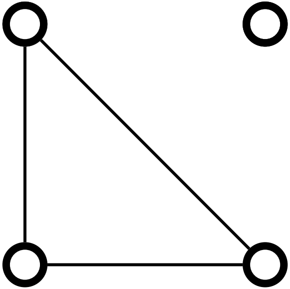

Algorithm of Tannier, Bergeron and Sagot

function PROCESS(graph G with no non-singleton all-white connected components)
    $S := ()$
    $G' := G$
    while there is a black node v in G' **do**
        apply toggle(v) to G'
        $S := S, v$
    while there is a non-singleton all-white connected component in G/S **do**
        $U :=$ set of nodes from non-singleton all-white connected components in G/S
        $S_1, S_2 := S$ for the longest S_1 s.t. G/S_1 has a node of U in a not-all-white component
        $G_1 := G/S_1$
        $S_3 := ()$
        while there is a black node v from U in G_1 **do**
            apply toggle(v) to G_1
            $S_3 := S_3, v$
        if $S_2[1]$ is white in G_1 **then**
            remove the last element w from S_3
            undo toggle(w) in G_1
        $S := S_1, S_3, S_2$
    return S

Interface of Tannier, Bergeron and Sagot

We need to efficiently maintain $\pi \in S_n$ and a set $V \subseteq [n]$ under the following operations:

1. query for π_i or π_i^{-1},
2. apply to π a signed reversal of a given interval,
3. find $i \in V$ such that i and $i+1$ have different signs in π,
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$\mathcal{O}(\log n)$ time: splay tree + reverse flag

Operations 3 and 4 were previously implemented in $\mathcal{O}(\sqrt{n})$ time.

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Red-blue graph

For permutation $\pi = (0, -2, -5, 1, -4, 6, -3, 7)$ add the following nodes and blue edges:

$$0 - 2 - 5 - 1 - 4 - 6 - 3 - 7$$

$$0' - 2' - 5' - 1' - 4' - 6' - 3' - 7'$$

Add red edges $\{i, (i+1)\}$ and $\{i', (i+1)'\}$ if i and $i+1$ have the same sign in π and $\{i, (i+1)'\}$ and $\{i', (i+1)\}$ otherwise.

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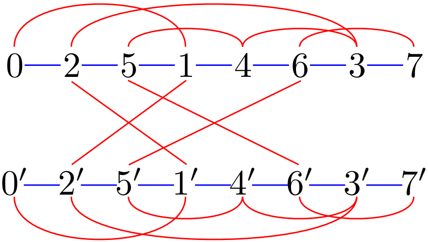
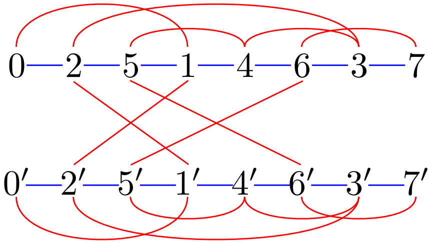


Reversal of an interval

Signed reversal of an interval can be simulated by reattaching 4 blue edges, e.g. consider reversing $[2, 4]$ on $\pi = (0, -2, \underline{-5, 1, -4}, 6, -3, 7)$:



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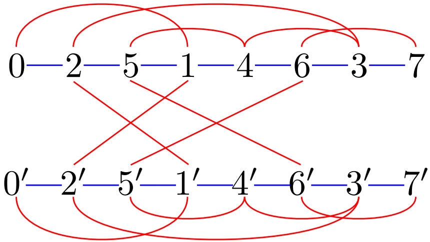

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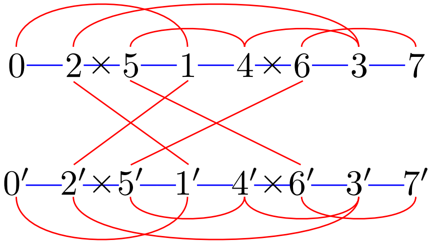

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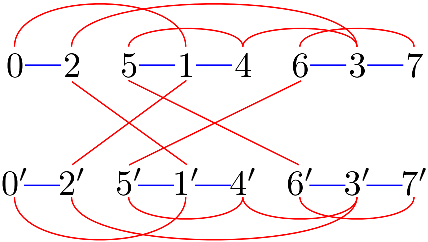

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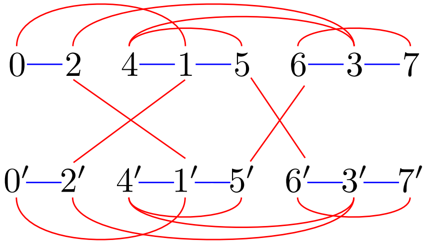

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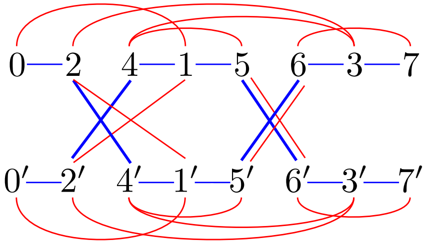

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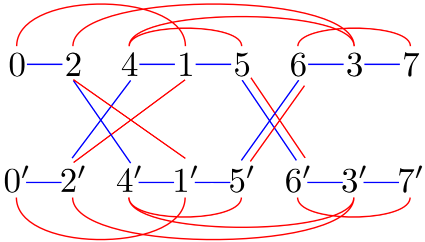

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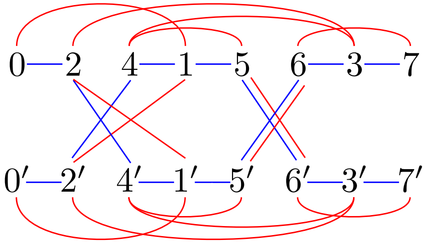
Properties of the graph



Property

Node *i* is in the blue component of 0 if and only if *i* has the same sign in π as in π_0.

Corollary

The endpoints of a red edge $i-(i+1)/(i+1)'$ are in distinct blue components iff *i* and $i+1$ have different signs in π.

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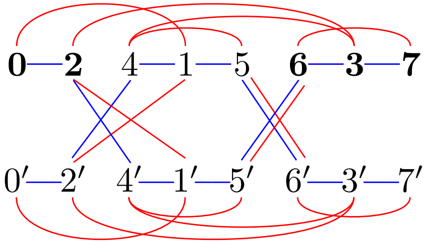

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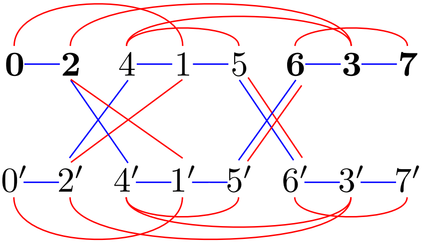

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Reformulation of the problem

New goal

Find a red edge connecting distinct blue components under insertions and deletions of blue edges and deletions of red edges.

Algorithm

Put weights on edges:

- 0 on blue edges
- i on i-th red edge

If the graph is connected:

$\Longrightarrow$ its MST has exactly one red edge

$\Longrightarrow$ weight of MST gives the index of the red edge.

Use the algorithm by Holm, Lichtenberg and Thorup for dynamic MST in amortized $\mathcal{O}(\log^4 n)$ time.

$\Longrightarrow$ $\mathcal{O}(n \log^4 n)$ total time.

Reformulation of the problem

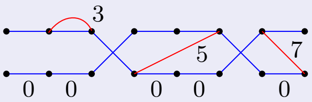
New goal

Find a red edge connecting distinct blue components under insertions and deletions of blue edges and deletions of red edges.

Algorithm

Put weights on edges:

- 0 on blue edges
- *i* on *i*-th red edge



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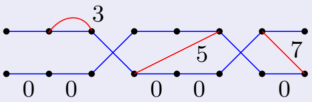

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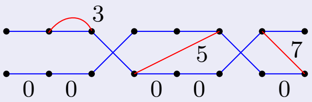

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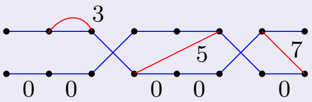

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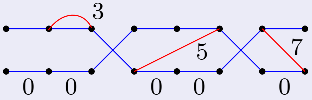

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Faster approach

Instead of MST: maintain a spanning forest of the graph on blue and red edges in a link-cut tree in amortized $\mathcal{O}(\log^2 n)$ time (using a data structure for fully-dynamic graph connectivity).

How to find a red edge connecting blue components of 0 and $0'$?

Recall:

1. Node i is in the blue component of 0 if and only if i has the same sign in π as in π_0.
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Solution: binary search over the path connecting 0 and $0'$!

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In the previous approach:

1. $\mathcal{O}(n \log n)$ queries about the sign of i in π
2. $\mathcal{O}(n)$ updates

Idea: maintain shortcuts of length $\varepsilon \log \log n$ up the splay trees

1. $\mathcal{O}(\log n / \log \log n)$ time per query
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Open questions

1. Can we improve the time complexity even further?
   – Our solution runs in $\mathcal{O}(n \log^2 n / \log \log n)$ but counting the numer of reversals takes $\mathcal{O}(n)$ time
   – Swenson [arXiv '24]: $\mathcal{O}(n \log n)$
2. Does randomization help?
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