Measure Properties of Regular Sets of Trees

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Abstract. We investigate measure theoretic properties of regular sets of infinite trees. As a first result, we prove that every regular set is universally measurable and that every Borel measure on the Polish space of trees is continuous with respect to a natural transfinite stratification of regular sets into ω_1 ranks. We also expose a connection between regular sets and the σ-algebra of $\mathcal{R}$-*sets*, introduced by A. Kolmogorov in 1928 as a foundation for measure theory. We show that the *game tree languages* $\mathcal{W}_{i,k}$ are Wadge-complete for the finite levels of the hierarchy of $\mathcal{R}$-sets. We apply these results to answer positively an open problem regarding the game interpretation of the probabilistic μ-calculus.

1 Introduction

Among logics for expressing properties of concurrent processes, represented as nondeterministic transition systems (NTS's), Rabin's Monadic Second Order Logic [33] and Kozen's modal μ-calculus [26] play a fundamental rôle. The two logics are closely related (see, e.g. [19]) and enjoy an intimate connection with parity games [11,19,36]. An abstract setting for investigating topological properties of regular sets, using the tools of descriptive set theory, is given by so-called *game tree languages* of [1] (see also [2]). For natural numbers $i < k$, the language $\mathcal{W}_{i,k}$ is the regular set of parity games with priorities in $\{i \dots k\}$, played on an infinite binary tree structure, which are winning for Player $\exists$. The (i,k)-indexed sets $\mathcal{W}_{i,k}$ form a strict hierarchy of increasing topological complexity called the *index hierarchy* of game tree languages [1,2,7]. Precise definitions are presented in Section 2.

For many purposes in computer science, it is useful to add probability to the computational model, leading to the notion of probabilistic nondeterministic transition systems (PNTS's). In an attempt to identify a satisfactory analogue of Kozen's μ-calculus for expressing properties of PNTS's, the third author has

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recently introduced in [31,32] a quantitative fixed-point logic called *probabilistic μ-calculus with independent product* (pLμ). A central contribution of [32] is the definition of a game interpretation of pLμ, given in terms of a novel class of games generalizing ordinary two-player *stochastic* parity games. While in ordinary two-player (stochastic) parity games the outcomes are infinite sequences of game-states, in pLμ-games the outcomes are infinite trees, called *branching plays*, whose vertices are labelled with game-states. This is because in pLμ games some of the game-states, called *branching states*, are interpreted as generating distinct game-threads, one for each successor state of the branching state, which continue their execution *concurrently* and *independently*. The winning set of a pLμ-game is therefore a collection of branching plays specified by a combinatorial condition associated with the structure of the game arena.

Unlike winning sets of ordinary two-player (stochastic) parity games, which are well-known to lie in the $\mathbf{\Delta}^0_3$ class of sets in Borel hierarchy, the winning sets of pLμ-games generally belong to the $\mathbf{\Delta}^1_2$-class of sets in the projective hierarchy of Polish spaces [32, Theorem 4.20]. This high topological complexity is a serious concern because pLμ-games are *stochastic*, i.e. the final outcome (the branching play) is determined not only by the choices of the two players but also by the randomized choices made by a probabilistic agent. A pair of strategies for $\exists$ and $\forall$, representing a play up–to the choice of the probabilistic agent, only defines a probability measure on the space of outcomes. For this reason, one is interested in the *probability* of a play to satisfy the winning condition. Under the standard Kolmogorov's measure-theoretic approach to probability theory, a set has a well-defined probability only if it is a *measurable*[4] *set*. Due to a result of Kurt Gödel (see [20, §25]), it is consistent with Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) that there exists a $\mathbf{\Delta}^1_2$ set which is not measurable. This means that it is not possible to prove (in ZFC) that all $\mathbf{\Delta}^1_2$-sets are measurable. However it may be possible to prove that a *particular* set (or family of sets) in the $\mathbf{\Delta}^1_2$-class is measurable. In [31] the author asks the following question[5]: are the winning sets of pLμ-games measurable in ZFC? As already observed in [31, §5.4], the problem can be equivalently reformulated, using well-known concepts and terminology, as follows:

Question 1. Are the game tree languages $\mathcal{W}_{i,k}$ measurable in ZFC?

A positive answer to Question 1 implies, as an immediate corollary, that every regular set of trees (i.e. definable in Rabin's Monadic Second Order Logic [33]) is measurable. This follows from the fact that continuous pre-images of universally measurable sets are universally measurable (cf. Proposition 1 in Section 2).

In his work on the probabilistic μ-calculus [31,32], the third author introduced a method for evaluating the probability of sets of branching plays. Once rephrased in the terminology of game tree languages, the method consists in a

[4] In this paper the adjective measurable always stands for *universally measurable*, see Section 2 for definitions.

[5] Statement "is mG-UM($\mathit{\Gamma}_p$) true?", see Definition 5.1.18 and discussion at the end of Section 4.5 in [31]. See also Section 8.1 in [32].

transfinite characterization of $\mathcal{W}_{i,k}$ as the union of a chain of *simpler* subsets $\mathcal{W}^{\alpha}_{i,k}$, indexed by countable ordinals $\alpha < \omega_1$, in such a way that

$$\mathcal{W}_{i,k} = \bigcup_{\alpha<\omega_1} \mathcal{W}^{\alpha}_{i,k}.$$

Precise definitions are given in Section 2.1. This is used in [31,32] to evaluate the probability $\mu(W_{i,k})$ in terms of the limit of the probabilities of its simpler approximants using the equality

$$\mu(\mathcal{W}_{i,k}) = \sup_{\alpha<\omega_1} \mu(\mathcal{W}^{\alpha}_{i,k}).$$

This equality, however, expresses a form of $\aleph_1$-continuity of the measure μ which does not follow from the standard properties of measures which are only σ-continuous. For this reason the author asks:

Question 2. Does $\mu(\mathcal{W}_{i,k}) = \sup_{\alpha<\omega_1} \mu(\mathcal{W}^{\alpha}_{i,k})$ hold for all Borel probability measures μ?

It was observed in [31,32] that both Questions 1 and Question 2 can be proved in ZFC + $\mathrm{MA}_{\aleph_1}$, the extension of ZFC with "Martin's Axiom at $\aleph_1$". Indeed ZFC + $\mathrm{MA}_{\aleph_1}$ proves that every $\mathbf{\Sigma}^1_2$ set is universally measurable (this solves Question 1 since $\mathcal{W}_{i,k}$ belong to $\mathbf{\Delta}^1_2 \subseteq \mathbf{\Sigma}^1_2$) and that, for every Borel measure μ, the equality $\mu(\bigcup_\alpha X^\alpha) = \sup_\alpha \mu(X_\alpha)$ holds for arbitrary ω_1-indexed collections X_α of measurable sets (this solves Question 2 taking $X_\alpha = \mathcal{W}^{\alpha}_{i,k}$). For more informations regarding Martin's Axiom, see [14].

1.1 Main contribution

We succeeded to solve Questions 1 and 2 originally motivating this work.

Theorem 1. *For every $i < k$ the game tree language $\mathcal{W}_{i,k}$ is measurable.*

Theorem 2. *For every $i < k$, with k odd, and for every Borel measure μ on $\mathrm{Tr}_{i,k}$ the following equality holds:*

$$\mu(\mathcal{W}_{i,k}) = \sup_{\alpha<\omega_1} \mu(\mathcal{W}^{\alpha}_{i,k}).$$

We provide in Section 3 a self-contained ZFC proof of both theorems. We use a method of Lusin and Sierpiński [28] originally applied to prove measurability of analytic sets and later applied by A. Kolmogorov [35] to prove measurability of $\mathcal{R}$-sets (discussed below).

Together, our positive answers to Question 1 and Question 2 imply that the results of [31,32] about the game semantics of the probabilistic μ-calculus hold in ZFC alone.

1.2 Kolmogorov's $\mathcal{R}$-sets

Before discovering that the proof method of Lusin and Sierpiński could be applied to solve both questions from [31,32], we found another interesting way to give a positive answer to Question 1 in ZFC alone. This resulted in the discovery of a tight connection between the notion of $\mathcal{R}$-sets, introduced below in this introduction and discussed in details in Section 4, and the combinatorial machinery of parity games.

Measure theoretic problems such as the one formulated in Question 1 have been investigated since the first developments of measure theory, in late 19th century, as the existence of non-measurable sets (e.g. Vitali sets [20]) was already known. The measure-theoretic foundations of probability theory are based around the concept of a σ-algebra of measurable events on a space of potential outcomes. Typically, the σ-algebra is assumed to contain all open sets. Hence the minimal σ-algebra under consideration consists of all Borel sets whereas the maximal consists, by definition, of the collection of all measurable sets. The Borel σ-algebra, while simple to work with, lacks important classes of measurable sets such as the *analytic* ($\mathbf{\Sigma}^1_1$) sets. On the other hand, the full σ-algebra of measurable sets may be difficult to work with since there is no constructive methodology for establishing its membership relation, i.e. for proving that a given set belongs to this σ-algebra. This picture led to a number of attempts to find larger σ-algebras, extending the Borel σ-algebra and including as many measurable sets as possible and, at the same time, providing practical techniques for establishing the membership relation.

A classical methodology for constructing such σ-algebras is to identify a family $\mathcal{F}$ of "safe" operations on sets which, when applied to measurable sets are guaranteed to produce measurable sets. When the operations considered have countable arity (e.g. countable union), the σ-algebra generated by the open sets closing under the operations in $\mathcal{F}$ admits a transfinite decomposition into ω_1 levels, and this allows the membership relation to be established inductively. The simplest case is given by the σ-algebra of Borel sets, with $\mathcal{F}$ consisting of the operations of complementation and countable union. Other less familiar examples include $\mathcal{C}$-sets studied by E. Selivanovski [34], Borel programmable sets proposed by D. Blackwell [6] and $\mathcal{R}$-sets proposed by A. Kolmogorov [25].

Most measurable sets arising in ordinary mathematics are $\mathcal{R}$-sets belonging to the finite levels of the transfinite hierarchy of $\mathcal{R}$-sets. For example, all Borel sets, analytic sets, co-analytic sets and Selivanovski's $\mathcal{C}$-sets lie in the first two levels [10]. Furthermore, the inductive proof method for establishing membership in the class of $\mathcal{R}$-sets has allowed the development of a rich theory of $\mathcal{R}$-sets. Beside the original work of Kolmogorov [25], fundamental results were obtained by Lyapunov [29] and, more recently, by Burgess [10]. Further progress can be found in the work of Barua [3,4]. The basic definitions on $\mathcal{R}$-sets are presented in Section 4. We refer to [22] for a modern introduction to the subject.

As a main technical contribution of this paper we prove the following theorem in Section 5.

Theorem 3. *$\mathcal{W}_{k-1,2k-1}$ is complete for the k-th level of the hierarchy of $\mathcal{R}$-sets.*

In particular, game tree languages $\mathcal{W}_{i,k}$ are $\mathcal{R}$-sets and therefore measurable. Thus Theorem 3 provides an alternative proof of Question 1. Furthermore, the theory of $\mathcal{R}$-sets sheds some additional light on the properties of game tree languages and regular sets. For example, a basic result of this theory (see, e.g. [4, Theorem 2.8]) states that every $\mathcal{R}$-set has the Baire property. Hence $\mathcal{W}_{i,k}$, and thus every regular set of trees, have the Baire property.

The result of Theorem 3 also contributes to the abstract theory of $\mathcal{R}$-sets. Indeed, to the best of our knowledge, the game tree languages $\mathcal{W}_{i,k}$ are the first natural example of sets complete for the finite levels of the $\mathcal{R}$-hierarchy. Having examples of complete sets sheds additional light on the concept of $\mathcal{R}$-sets and, in analogy with the study of complexity classes in computational complexity theory, may simplify further investigations.

Another interesting aspect of our work is the following. The proof of Theorem 3 is obtained by first introducing in Section 5 a novel class of sets definable by *parametrized parity games* which we call *Matryoshka games*. The usefulness of this notion comes from the following observations:

i) every $\mathcal{R}$-set belonging to the k-th level of the $\mathcal{R}$-hierarchy can be defined by a Matryoshka game using priorities in the range $(k, 2k-1)$,
ii) the game tree language $\mathcal{W}_{i-1,k}$ is a complete set among the sets definable by Matryoshka games with priorities in (i, k).

These two observations imply that the game tree language $\mathcal{W}_{k-1,2k-1}$ is *hard* for the k-level of the $\mathcal{R}$-hierarchy. Then the result of Theorem 3, establishing membership of $\mathcal{W}_{k-1,2k-1}$ in the k-th level of the $\mathcal{R}$-hierarchy, completes the picture.

As a consequence, the class of Matryoshka definable sets and the class of sets belonging to the finite levels of the $\mathcal{R}$-hierarchy, coincide. This indicates that the combinatorics introduced by Kolmogorov for defining a large σ-algebra of measurable sets and that of parity games, developed since the 80's in Computer Science to investigate ω-regular properties of transition systems, are closely related. It is suggestive to think that the origins of the concept of parity games could be backdated to the original work of A. Kolmogorov.

1.3 Boundedness principle.

In attempts to solve Question 2, that is the problem of establishing the $\aleph_1$-continuity property, we also tried a very natural approach based on the *Boundedness Principle* (see, e.g. Section 34.B in [24]).

We discovered that this approach solves the problem of $\aleph_1$-continuity in the simplest case of $\mathcal{W}_{0,1}$. We discuss this argument in Section 6 along with a counter-example showing that this method does not generalize to $\mathcal{W}_{1,3}$ or higher indices.

1.4 Related work.

A game-theoretic approach to $\mathcal{R}$-sets, closely related to this work, is developed by Burgess in [10] where the following characterization is stated as a remark without a formal proof: (1) every set $A \subseteq X$ belongs to a finite level of the hierarchy of $\mathcal{R}$-sets if and only if it is of the form $A = \Game(K)$, for some set $K \subseteq \omega^\omega$ which is a Boolean combination of F_σ sets, and (2) the levels of the hierarchy of $\mathcal{R}$-sets are in correspondence with the levels of the *difference hierarchy* (see [24, §22.E]) of F_σ sets. The operation $\Game$ is the so-called *game quantifier* (see [24, §20.D] and [8,9,21,30]). Admittedly, our characterization of $\mathcal{R}$-sets in terms of Matryoshka games, can be considered as a modern variant of the result of Burgess. From the theorem of Burgess one can relatively easily infer Theorem 1 through appropriately formulated reductions. Since we were interested in proving both Theorems 1 and Theorem 2, we decided to reconstruct the argument of Burgess in the terminology of Matryoshka games in order to investigate in a more convenient framework the issue of $\aleph_1$-continuity.

While we did not manage to solve the $\aleph_1$-continuity problem from this approach, we discovered interesting new information about game tree languages $\mathcal{W}_{i,k}$, and thus about regular languages of trees, as formulated in Theorem 3.

In another direction, a result of Fenstand and Normann [13], which builds on previous work of Solovay, can be used to give a very succinct proof of Theorem 1, that is a proof of the measurability of $\mathcal{W}_{i,k}$. In their paper [13], the authors introduce the class of *absolutely* $\mathbf{\Delta}^1_2$ *sets* contained in $\mathbf{\Delta}^1_2$ and consisting only of measurable sets. The measurability of $\mathcal{W}_{i,k}$ then follows from the observation that $\mathcal{W}_{i,k}$ is an absolutely $\mathbf{\Delta}^1_2$ set. This fact was already noticed in the proof of Theorem 6.6 in [17] and was exploited to establish that all regular sets (and thus also $\mathcal{W}_{i,k}$) are Baire measurable. As we already observed earlier, our proof that game tree languages $\mathcal{W}_{i,k}$ are $\mathcal{R}$-set gives an alternative proof of Baire measurability of regular languages.

The method of absoluteness is very general and can be arguably used to settle virtually all measurability questions arising in ordinary mathematics. However, at the moment of writing of this article, the authors don't see how to use this approach to give an alternative proof of Theorem 2 stated above.

1.5 Some Further Remarks

This article builds on the results of [15] announced at the annual MFCS 2014 conference in Budapest. In [15] the authors provided a positive answer to Question 1 of [31,32] by establishing the connection with the theory of $\mathcal{R}$-sets by proving Theorem 3. We also announced, without proofs, that Question 2 of [31,32] could be given a positive answer assuming the (not provable in ZFC alone) determinacy of so-called Harrington games. Only after the submission, we discovered that the method of Lusin and Sierpiński could be used to answer both Question 1 and Question 2, and that the the result of Fenstand and Normann could also give a very short proof of Question 1.

1.6 Organization of the Paper

The rest of the paper is organized as follows. In Section 2 we provide the necessary basic definitions of descriptive set theory and about regular languages on trees, including the stratification of $\mathcal{W}_{i,k}$ into ω-levels $\mathcal{W}^{\alpha}_{i,k}$. In Section 3 we prove Theorems 1 and 2 by applying the method of Lusin and Sierpiński. In Section 4 we provide the basic definitions of the theory of $\mathcal{R}$-sets. In Section 5 we introduce Maytroshka games and prove Theorem 3. In Section 6 we discuss the applicability of the Boundedness principle to solve the $\aleph_1$-continuity problem of $\mathcal{W}_{0,1}$.

2 Basic Notions from Descriptive Set Theory

We assume the reader is familiar with the basic notions of descriptive set theory and measure theory. We refer to [24] as a standard reference on these subjects.

Given two sets X and Y, we denote with X^Y the set of functions from Y to X. We denote with 2 and ω the two element set and the set of all natural numbers, respectively. The powerset of X will be denoted by both 2^X and $\mathcal{P}(X)$, as more convenient to improve readability. A topological space is *Polish* if it is separable and the topology is induced by a complete metric. A set is *clopen* if it is both closed and open. A space is *zero-dimensional* if the clopen subsets form a basis of the topology. In this work we limit our attention to zero-dimensional Polish spaces. Let X, Y be two topological spaces and $A \subseteq X$, $B \subseteq Y$ be two sets. We say that A is *Wadge reducible* to B, written as $A \leq_W B$, if there exists a continuous function $f\colon X \to Y$ such that $A = f^{-1}(B)$. Two sets A and B are *Wadge equivalent* (denoted $A \sim_W B$) if $A \leq_W B$ and $B \leq_W A$ hold. Given a family $\mathcal{C}$ of subsets of X, we say that a set $A \subseteq \mathcal{C}$ is *hard for* $\mathcal{C}$ if $B \leq_W A$ holds for all $B \in \mathcal{C}$. The set A is *complete for* $\mathcal{C}$ if it is hard and $A \in \mathcal{C}$.

Given a Polish space X, we denote with $\mathcal{M}_{=1}(X)$ the Polish space of all Borel probability measures μ on X (see e.g. [24, Theorem 17.22]). A set $N \subseteq X$ is *μ-null* if there exists a Borel set $B \supseteq N$ such that $\mu(B) = 0$. A set $A \subseteq X$ is μ-measurable if $A = B \cup N$, for a Borel set B and a μ-null set N. A set $A \subseteq X$ is *universally measurable* if it is μ-measurable for all $\mu \in \mathcal{M}_{=1}(X)$. In what follows we omit the "universally" adjective. Measurable sets are closed under taking continuous pre-images (see, e.g. Corollary 7.44.1 in [5]).

Proposition 1. *If $A \leq_W B$ and B is measurable then A is measurable.*

Given a finite alphabet Σ, we denote with Tr_Σ the collection $\Sigma^{\{0,1\}^*}$ of labellings of the vertices $\{0,1\}^*$ of the full binary tree with elements of Σ. The set Tr_Σ is endowed with the standard Polish topology (see e.g. [2]) so that Tr_Σ is homeomorphic to the *Cantor space* 2^ω.

Given two natural numbers $i < k$, we succinctly denote with $\mathrm{Tr}_{i,k}$ the space Tr_Σ with $\Sigma = \{\exists, \forall\} \times \{i, \ldots, k\}$. Each $t \in \mathrm{Tr}_{i,k}$ can be interpreted as a two-player parity game with priorities in $\{i, \ldots, k\}$, with players $\exists$ and $\forall$ controlling vertices labelled by $\exists$ and $\forall$, respectively. As usual we consider the standard formulation of parity games, where a play is winning for $\exists$ if and only if the greatest priority visited infinitely often if even.

Definition 1 ([2]). *Given two natural numbers $i < k$, the* game tree language $\mathcal{W}_{i,k}$ *is the subset of* $\mathrm{Tr}_{i,k}$ *consisting of all parity games admitting a winning strategy for* $\exists$*. The pair* (i,k) *is called the* index *of* $\mathcal{W}_{i,k}$*.*

Clearly, there is a natural Wadge equivalence between the languages $\mathcal{W}_{i,k}$ and $\mathcal{W}_{i+2,k+2}$. Therefore, we identify indices (i,k) and $(i+2j, k+2j)$ for every $i \leq k$ and $j \in \omega$. Indexes can be partially ordered by defining $(i,k) \subseteq (i',k')$ if and only if $\{i,\ldots,k\} \subseteq \{i',\ldots,k'\}$.

It is well known that, for every $i < k$ the language $W_{i,k}$ is *regular*, i.e. definable in Rabin's Monadic Second Order Logic on the full binary tree [33]. The importance of game tree languages in the study of regular languages of trees is expressed by the following proposition.

Proposition 2. *For any finite alphabet Σ and regular language $A \subseteq \mathrm{Tr}_\Sigma$ there exists an index (i,k) such that $A \leq_W W_{i,k}$.*

Proof. Let $\mathcal{A}$ be a parity tree automaton accepting the regular set A and let i and k be the lower and greatest priorities in $\mathcal{A}$, respectively. The automaton can be regarded as a transducer f continuously mapping Σ-labelled trees to parity games with priorities in $\{i,k\}$ in such a way that a tree t is accepted by $\mathcal{A}$ if and only if $f(t) \in W_{i,k}$. □

The following well-known result states that the hierarchy of game tree languages forms a chain of increasing topological complexity.

Theorem 4 ([2]). *If $(i,k) \subsetneq (i',k')$ then $\mathcal{W}_{i,k} \lneq_W \mathcal{W}_{i',k'}$.*

It is well known that the first level of this hierarchy, the languages $\mathcal{W}_{0,1}$ and $\mathcal{W}_{1,2}$, constitute examples of co-analytic ($\mathbf{\Pi}^1_1$) complete and analytic ($\mathbf{\Sigma}^1_1$) complete sets, respectively and that, for every $i < k$, the language $\mathcal{W}_{i,k}$ belongs to the $\mathbf{\Delta}^1_2$-class. Already at the second level, however, the regular tree languages $\mathcal{W}_{0,2}$ and $\mathcal{W}_{1,3}$ are not contained in the σ-algebra generated by the analytic sets ([18], see also [30]).

2.1 Ranks on Regular Tree Languages

In [31,32] the author investigated a transfinite inductive characterization of the game tree language $\mathcal{W}_{i,k}$, for $i < k$, whose general purpose is to describe $\mathcal{W}_{i,k}$ as a union of simpler sets. We recall this characterization in this section. Detailed informations can be found in Sections 4.3 and 6 of [31].

In what follows we restrict attention to indexes $i < k$ with k an odd number. All definitions and results below have their corresponding version for the case of k even, and are obtained by standard duality arguments (see, e.g. the end of Section 6.3 of [31]).

The intuitive idea of the construction can be understood as follows. In a game tree $t \in \mathrm{Tr}_{i,k}$ the maximal priority k is the most important. In particular, if a tree t does not contain occurrences of k, then t is "simple" as it already belongs to $\mathrm{Tr}_{i,k-1}$. Since k is odd, this is a role of $\exists$ to guarantee that every play visits only finitely many times the priority k. Therefore, we can approximate $\mathcal{W}_{i,k}$ by allowing more and more occurrences of the priority k. This motivates the following definition.

Definition 2 (Occurrences of k's). *Given a tree $t \in \mathrm{Tr}_{i,k}$ we say that a vertex $v \in \{0,1\}^*$ is an* occurrence of k *if $t(v)$ has priority k, i.e. if $t(v) = \langle \exists, k\rangle$ or $t(v) = \langle \forall, k\rangle$.*

We say that v is a first occurrence of k *if it is an occurrence of k and none of its nonempty predecessors (i.e. nodes v' such that $\epsilon \prec v' \prec v$) are occurrences of k.*

In the above definition the root of the tree is not considered as a first occurrence of k. The purpose of that will be explained in a moment.

For any tree $t \in \mathrm{Tr}_{i,k}$, the set of first occurrences of k forms an anti-chain of vertices in t. Thus t can be decomposed as depicted on Figure 1, where $v_0, v_1, \ldots$ is the (at most countable) collection of the first occurrences of k and $t_{v_i} = t{\restriction}_{v_i}$ is the subtree of t rooted at v_i.

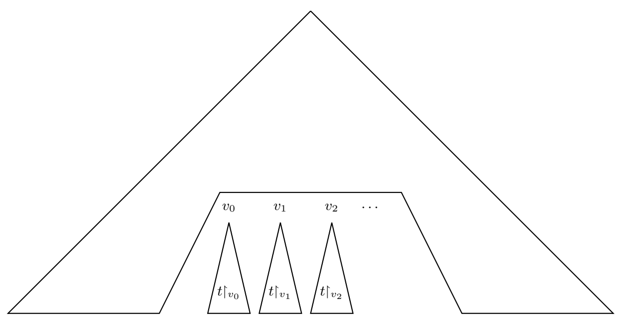

Fig. 1. A decomposition of a tree t as a substitution of subtrees $t{\restriction}_{v_i}$ into vertices v_i that are the first occurrences of k.

For the purpose of the analysis of the ranks, we assume that the alphabet $A_{i,k}$ is additionally equipped with symbols $\top$, $\bot$ that denote the winning positions — reaching a symbol $\top$ (resp. $\bot$) in the game on a given tree t makes the player $\exists$ (resp. $\forall$) win the play no matter what further symbols are visited.

Let $\top$ (resp. $\bot$) stand for the tree labelled everywhere by $\top$ (resp. $\bot$). By the definition we have that $\top \in \mathcal{W}_{i,k}$ and $\bot \notin \mathcal{W}_{i,k}$.

Assume that $X \subseteq \mathrm{Tr}_{i,k}$ is a set of trees. Intuitively X stands for the set of trees on which $\exists$ can win and guarantee herself to visit only few occurrences of k. Our aim is to define $\pi_X(t)$ as a tree where instead of each first occurrence v_i of k, a subtree $\top$ or $\bot$ is plugged, depending whether $t{\restriction}_{v_i} \in X$ or not. More formally consider $t \in \mathrm{Tr}_{i,k}$ with the first occurrences of k being $v_0, \ldots$. For every such occurrence v_i define t_i as $\top$ if $t{\restriction}_{v_i} \in X$ and $\bot$ otherwise. Let t' be the same tree as t but with priority i set in the root. Now let $\pi_X(t)$ be obtained by plugging trees t_i as subtrees under nodes v_i in the tree t', such an operation is denoted as follows:

$$\pi_X(t) = t'\big[v_i \leftarrow t_i\big]_{i=0,\dots}.$$

Observe that $\pi_X(t) \in \mathrm{Tr}_{i,k-1}$ as the only occurrences of k in t are in subtrees substituted by $\top$, $\bot$, and possibly in the root.

Definition 3. *We define the operator $\mathcal{W}\colon \mathcal{P}(\mathrm{Tr}_{i,k}) \to \mathcal{P}(\mathrm{Tr}_{i,k})$ as follows:*

$$\mathcal{W}(X) = \big\{t \mid \pi_X(t) \in \mathcal{W}_{i,k-1}\big\}$$

The following basic properties of $\mathcal{W}$, which are folklore, are listed in the following lemma (see, e.g. Lemmas 6.2.15–16 in [31] for detailed proofs).

Lemma 1. *The following assertions hold:*

1. *(monotonicity) If $X \subseteq Y$ then $\mathcal{W}(X) \subseteq \mathcal{W}(Y)$,*
2. *(fixed-point) $\mathcal{W}(\mathcal{W}_{i,k}) = \mathcal{W}_{i,k}$*

Now we can formally define the approximants of $\mathcal{W}_{i,k}$.

Definition 4. *Let us define by transfinite induction:*

$$\begin{aligned} \mathcal{W}^0_{i,k} &= \emptyset \\ \mathcal{W}^{\alpha+1}_{i,k} &= \mathcal{W}\big(\mathcal{W}^{\alpha}_{i,k}\big) \\ \mathcal{W}^{\beta}_{i,k} &= \bigcup_{\alpha<\beta} \mathcal{W}^{\alpha}_{i,k} \qquad\qquad \textit{(for } \beta \textit{ limit ordinal)} \end{aligned}$$

The following result is proved as Lemmas 6.3.3 and 6.3.6 in [31].

Theorem 5. *The game tree language $\mathcal{W}_{i,k}$ is the least fixed-point of $\mathcal{W}$. This fixed-point is reached in ω_1 steps. In particular*

$$\mathcal{W}_{i,k} = \bigcup_{\alpha<\omega_1} \mathcal{W}^{\alpha}_{i,k}.$$

Definition 5. *A tree $t \in \mathcal{W}_{i,k}$ is said to have* rank α *if α is the least ordinal such that $t \in \mathcal{W}^{\alpha}_{i,k}$. Note that every t has a countable rank.*

3 Main result using the Lusin–Sierpiński method

The two measurability questions regarding regular tree languages, left open in [31,32] and stated as Question 1 and Question 2 in the introduction, can be formally stated as follows.

Theorem 1. *For every $i < k$ the game tree language $\mathcal{W}_{i,k}$ is measurable.*

Note that the case for $k-i=1$ is trivial. Indeed $\mathcal{W}_{1,2}$ is a well-known example of analytic ($\boldsymbol{\Sigma}^1_1$) complete set and every such set is measurable (see, e.g. [24, §29.B]). Already for $\mathcal{W}_{0,2}$, however, measurability is not obvious since $\mathcal{W}_{0,2}$ is not contained in the σ-algebra generated by the analytic sets (see Remarks after Theorem 4).

Theorem 2. *For every $i<k$, with k odd, and for every Borel measure μ on $\mathrm{Tr}_{i,k}$ the following equality holds:*

$$\mu(\mathcal{W}_{i,k}) = \sup_{\alpha<\omega_1} \mu(\mathcal{W}^{\alpha}_{i,k}).$$

Clearly the statement of Theorem 2 does not follow from σ-continuity of measures because the supremum is taken over an uncountable chain of sets.

In this section we adapt a proof method of N. Lusin and W. Sierpiński introduced in [27] which allows for a uniform proof of both Theorem 1 and Theorem 2. Originally the method has been applied to prove measurability of analytic sets.

In what follows, we fix an arbitrary pair $i<k$. We first introduce a notion of *j-schema* in Section 3.1. Section 3.2 shows that j-schema admit properties of duality and locality. In Section 3.3 we prove the crucial result — that the operations defined by j-schemas preserve measurability. It will directly imply Theorem 1. Finally, in Section 3.4 we show how to obtain Theorem 2 using the presented construction.

3.1 j-schemas

We start by defining a variant of the operation $\mathcal{W}(X)$ from Definition 3 that will allow us to prove our results inductively.

Definition 6. *Assume that $i-1\leq j\leq k$. A j-*schema *is a tuple*

$$\big(R^{(j+1)}_v,\ldots,R^{(k)}_v\big)_{v\in\{0,1\}^*},$$

where all the sets $R^{(p)}_v$ are subsets of $\mathrm{Tr}_{i,k}$.

Intuitively, $R^{(p)}_v$ contains trees on which $\exists$ should instantly win from the node v if this node has priority p. The crucial difference with the set X from Definition 3 is that all the trees in sets $R^{(p)}_v$ are rooted in the same node — we do not restrict to the subtree $t{\restriction}_v$.

Definition 7. *Given a j-schema $\mathcal{S}=\big(\boldsymbol{R_v}\big)_v$, a tree $t\in\mathrm{Tr}_{(i,k)}$, and two vertices $v\prec w\in\{0,1\}^*$, we say that w is a* victory from *v if the priority p of $t(w)$ is greater than j.*

A victory w from v is an $\exists$-victory if $t\in R^{(p)}_w$. A victory w is an $\forall$-victory otherwise.

Since we require that $v\prec w$, no matter what is the priority of $t(v)$, the node $w=v$ cannot be a victory from v. Intuitively it means that the priority of $t(v)$ does not influence victories from v, and corresponds to the assumption that ϵ is never a first occurrence of k in a tree t.

Definition 8. *Given a tree t and a vertex $w \in \{0,1\}^*$ which is a victory from v (either $\exists$ or $\forall$) we say that w is a* first victory *if no predecessor of w (i.e. a node w' such that $v \prec w' \prec w$) is a victory from v.*

In an analogous way to the concept of "(first) occurrence of k" of Definition 2 in Section 2.1, given a tree t, the set of first victories from v forms an anti-chain of vertices w_i in t, so that we can consider a decomposition as depicted on Figure 1.

Definition 9. *Assume that $\mathcal{S}$ is a j-schema. Define the function $\pi_{\mathcal{S},v} \colon \mathrm{Tr}_{i,k} \to \mathrm{Tr}_{i,j}$ as follows. Consider a tree t with first victories from v being $w_0, \ldots$. For each such victory w_i define t_i as $\top$ if w_i is $\exists$-victory, otherwise $t_i = \bot$, where the trees $\top$ and $\bot$ are defined as in Section 2.1. Let t' be obtained from t by setting the priority in the node v as i. Now*

$$\pi_{\mathcal{S},v}(t) = t'\big[w_i \leftarrow t_i\big].$$

In the case when $i - 1 = j$, for every $v \prec w$ the node w is a victory from v. Therefore, in the tree $\pi_{\mathcal{S},v}(t)$, both subtrees under 0 and 1 are from $\{\top, \bot\}$. By bending the definition, we can assume that $\mathrm{Tr}_{i,j}$ in that case is the set of such trivial trees and $\mathcal{W}_{i,j}$ is the set of such trees where $\exists$ wins the finite duration game.

Remark 1. For $i < k$, the projection function π_X of Section 2.1 with parameter $X \subseteq \mathrm{Tr}_{i,k}$ coincides with the projection $\pi_{\mathcal{S},\epsilon}$ associated with the $(k-1)$-schema (i.e. $j = k - 1$) $\mathcal{S} = \big(R_v^{(k)}\big)_v$ where

$$R_v^{(k)} = \big\{t \mid t{\restriction}_v \in X\big\}.$$

The following definition is similar to Definition 3 of Section 2.1.

Definition 10. *We define:*

$$\mathcal{W}_{\exists,v}(\mathcal{S}) = \big\{t \mid \pi_{\mathcal{S},v}(t){\restriction}_v \in \mathcal{W}_{i,j}\big\}$$
$$\mathcal{W}_{\forall,v}(\mathcal{S}) = \big\{t \mid \pi_{\mathcal{S},v}(t){\restriction}_v \notin \mathcal{W}_{i,j}\big\}.$$

In other words (recalling the operation $\pi_{\mathcal{S},v}$) $\mathcal{W}_{P,v}(\mathcal{S})$ is the set of game trees t such that P has a strategy σ starting from v such that any play π consistent with σ:

- Either does not include any victory from v and is winning for the player P under the usual parity condition (in this case the priorities appearing in the play below v are between i and j),
- or it includes a (first) victory w from v which is a P-victory (see Definition 7).

Remark 2. Note that $\mathcal{W}_{i,k}$ can be defined as $\mathcal{W}_{\exists,\epsilon}(\mathcal{S})$ for the k-schema

$$()_{v\in\{0,1\}^*},$$

i.e. the schema not containing any sets $R_v^{(p)}$.

3.2 Duality and locality

It turns out that j-schemas are in a sense self-dual and local. In this subsection we describe these properties more formally.

Definition 11. *Given a tree $t \in \mathrm{Tr}_{i,k}$ we define its* dual *as the tree* $\mathrm{dual}(t) \in \mathrm{Tr}_{i+1,k+1}$ *defined as* $\mathrm{dual}(t)(v) = \langle \bar{P}, p+1 \rangle$ *if* $t(v) = \langle P, p \rangle$ *and* $P \in \{\exists, \forall\}$.

For a set $X \subseteq \mathrm{Tr}_{i,k}$ we denote with $\mathrm{dual}(X) = \{\mathrm{dual}(t) \mid t \in X\}$.

By the definition $t \in \mathcal{W}_{i,k}$ if and only if $\mathrm{dual}(t) \notin \mathcal{W}_{i+1,k+1}$.

Definition 12. *For a j-schema $\mathcal{S} = \big(R_v^{(j+1)}, \ldots, R_v^{(k)}\big)_v$ we define the* dual *$(j{+}1)$-schema*

$$\mathrm{dual}(\mathcal{S}) = \big(R_v'^{(j+2)}, \ldots, R_v'^{(k+1)}\big)_v$$

with $R_v'^{(p+1)} = \mathrm{dual}\big(\mathrm{Tr}_{i,k} \setminus R_v^{(p)}\big)$.

The following fact follows directly from definitions and determinacy of parity games [36].

Lemma 2. *The following equality holds:*

$$\mathcal{W}_{P,v}\big(\mathcal{S}\big) = \mathrm{Tr}_{(i,k)} \setminus \mathcal{W}_{\bar{P},v}\big(\mathcal{S}\big)$$

and, for $t \in \mathrm{Tr}_{i,k}$, it holds that:

$$t \in \mathcal{W}_{P,v}\big(\mathcal{S}\big) \Leftrightarrow \ \mathrm{dual}(t) \in \mathcal{W}_{\overline{P},v}\big(\mathrm{dual}(\mathcal{S})\big).$$

Also, j-schemas are *local*, as described by the following definition.

Definition 13. *Let $\mathcal{S}$, $\mathcal{S}'$ be two j-schemas and $t \in \mathrm{Tr}_{(i,k)}$ be a tree. We say that $\mathcal{S}$, $\mathcal{S}'$ are* t-equivalent *if for all sets $R_v^{(p)}$, $R_v'^{(p)}$ in $\mathcal{S}$, $\mathcal{S}'$, t belongs to $R_v^{(p)}$ if and only if t belongs to $R_v'^{(p)}$.*

The following lemma follows directly from definitions.

Lemma 3. *Given $t \in \mathrm{Tr}_{i,k}$ and two t-equivalent j-schemas $\mathcal{S}$, $\mathcal{S}'$ the following implications hold: $t \in \mathbb{W}_{P,v}(\mathcal{S}) \Leftrightarrow t \in \mathbb{W}_{P,v}(\mathcal{S}')$.*

3.3 Measurability

We are now finally ready to state the invariant of induction that will lead to the proof of Theorems 1 and 2.

Definition 14. *Given a Borel measure μ on $\mathrm{Tr}_{i,k}$, we say that a j-schema $\mathcal{S}$ is μ-measurable if all the sets in it are μ-measurable.*

Proposition 3. *For every Borel measure μ on $\mathrm{Tr}_{i,k}$ and μ-measurable j-schema $\mathcal{S}$ it holds that $\mathcal{W}_{P,v}(\mathcal{S})$ is μ-measurable (for all $P \in \{\exists, \forall\}$ and $v \in \{0,1\}^*$).*

Note that by Remark 2 it will directly imply Theorem 1. The rest of this section is devoted to proving this result. The proof goes by induction on the value of j, for $i-1 \leq j \leq k$.

Observe that in the case $i-1=j$ every vertex w such that $v \prec w$ is a victory. Therefore, whether a tree t belongs to $\mathcal{W}_{P,v}(\mathcal{S})$ depends only on the sets $\boldsymbol{R_w}$ and the label of t in w for $w = v, v0, v1$. Therefore, in this case the thesis of the proposition holds.

The duality properties listed in Lemma 2 allow us to reduce all the other cases to the case of $P = \exists$, j odd, and $i \leq j \leq k$. First, we observe that we can always exchange the players $P \leftrightarrow \bar{P}$. Now, given μ, $P \in \{\exists, \forall\}$, $i \leq j \leq k$, and j-schema $\mathcal{S}$ with j even we can apply the operation dual and solve the case for the measure dual(μ), $\bar{P}$, $i+1 \leq j+1 \leq k+1$ and the $(j{+}1)$-schema dual$(\mathcal{S})$.

Let us fix an odd j and assume, by inductive hypothesis, that the statement of the proposition holds for all j'-schemas with $j' < j$.

Let us fix a Borel measure μ on $\mathrm{Tr}_{i,k}$ and a μ-measurable j-schema

$$\mathcal{S} = \left(R_v^{(j+1)}, R_v^{(j+2)}, \ldots, R_v^{(k)}\right)_v.$$

Our aim is to prove that $\mathcal{W}_{\exists,v}(\mathcal{S})$ is μ-measurable for all v.

Let us define $E_v^0 = \emptyset$, for $v \in \{0,1\}^*$. We now construct inductively a family of sets E_v^α and use them define $(j{-}1)$-schemas:

$$\mathcal{S}^\alpha = \left(E_v^\alpha, R_v^{(j+1)}, R_v^{(j+2)}, \ldots, R_v^{(k)}\right)_v.$$

For the case α successor ordinal, we define

$$E_v^{\alpha+1} = \mathcal{W}_{\exists,v}\big(\mathcal{S}^\alpha\big).$$

For α limit ordinal we define $E_v^\alpha = \bigcup_{\beta<\alpha} E_v^\beta$.

Observe that by the inductive hypothesis on j we know that every E_v^α is μ-measurable, and therefore all $(j{-}1)$-schemas $\mathcal{S}^\alpha$ are μ-measurable.

Fact 1. *The sequences E_v^α, for every v, are increasing sequences of sets.*

Proof. It follows by induction from the definition of E_v^α. We know that $\emptyset = E_v^0 \subseteq E_v^1$ and if $E_v^\alpha \subseteq E_v^{\alpha+1}$ then $E_v^{\alpha+1} \subseteq E_v^{\alpha+2}$. □

Since no uncountable strictly increasing sequence of real numbers exists, for every $v \in \{0,1\}^*$ there exists a countable ordinal α_v such that the measure $\mu(E_v^\alpha)$ stabilizes at α_v, i.e.

$$\mu(E_v^{\alpha_v}) = \mu(E_v^\beta)\ , \text{ for all } \beta > \alpha_v.$$

Let $\alpha_\diamond$ be the supremum of α_v for $v \in \{0,1\}^*$. Note that $\alpha_\diamond$, being a limit of countably many countable ordinals, is itself a countable ordinal. Let

$$U = \bigcup_v \left(E_v^{\alpha_\diamond+1} \setminus E_v^{\alpha_\diamond}\right).$$

By the definition of $\alpha_\diamond$, we have that $\mu(E_v^{\alpha_\diamond+1} \setminus E_v^{\alpha_\diamond}) = 0$. Hence the set U is a countable union of μ-null sets and is therefore μ-null.

Our goal now is to prove that:

$$E_v^{\alpha_\diamond} \subseteq \bigcup_{\alpha<\omega_1} E_v^\alpha \subseteq E_v^{\alpha_\diamond} \cup U. \tag{1}$$

This will show that $\bigcup_\alpha E_v^\alpha$ is a measurable set, since every set contained between two measurable sets of equal measure (i.e. $E_v^{\alpha_\diamond}$ and $E_v^{\alpha_\diamond} \cup U$) is measurable.

The left containment is trivial, hence let's consider the inclusion $\bigcup_\alpha E_v^\alpha \subseteq E_v^{\alpha_\diamond} \cup U$ and assume, towards a contradiction, that there exists a tree $t \in \bigcup_\alpha E_v^\alpha$ such that $t \notin E_v^{\alpha_\diamond}$ and $t \notin U$.

The fact that $t \notin U$ implies, by definition of U, that $t \notin E_v^{\alpha_\diamond+1}$. Since by hypothesis $t \notin E_v^{\alpha_\diamond}$, this means that $\mathcal{S}^{\alpha_\diamond}$ and $\mathcal{S}^{\alpha_\diamond+1}$ are t-equivalent.

By Lemma 3 this implies that $t \in \mathbb{W}_{v,P}(\mathcal{S}^{\alpha_\diamond}) \Leftrightarrow t \in \mathbb{W}_{v,P}(\mathcal{S}^{\alpha_\diamond+1})$. By the definition of E_v^α, for any α, this means that $t \in E_v^{\alpha_\diamond+1}$ if and only if $t \in E_v^{\alpha_\diamond+2}$.

Hence $\mathcal{S}^{\alpha_\diamond+1}$ and $\mathcal{S}^{\alpha_\diamond+2}$ are t-equivalent. By iterating the process, all the $(j-1)$-schemas $\mathcal{S}^\beta$ for $\beta \geq \alpha_\diamond$ are t-equivalent. In particular, $t \notin \bigcup_\beta E_v^\beta$ because $t \notin E_v^{\alpha_\diamond}$. A contradiction.

What remains is to prove a relation between the sets E_v^α and $\mathcal{W}_{i,j}^\alpha$ as expressed by the following lemma.

Lemma 4. *For every $v \in \{0,1\}^*$ and $\alpha < \omega_1$ we have*

$$E_v^\alpha = \big\{t \mid \pi_{\mathcal{S},v}(t){\restriction_v} \in \mathcal{W}_{i,j}^\alpha\big\}. \tag{2}$$

Proof. The proof is inductive on α. For $\alpha = 0$ and α limit the equality holds by the definition. Assume that the equality holds for α and all v, we need to prove it for $\alpha + 1$.

Unrevealing the definitions we obtain:

$$t \in E_v^{\alpha+1} \Leftrightarrow \pi_{\mathcal{S}^\alpha,v}(t){\restriction_v} \in \mathcal{W}_{i,j-1}$$
$$\pi_{\mathcal{S},v}(t){\restriction_v} \in \mathcal{W}_{i,j}^{\alpha+1} \Leftrightarrow \pi_{\mathcal{W}_{i,j}^\alpha}\big(\pi_{\mathcal{S},v}(t){\restriction_v}\big) \in \mathcal{W}_{i,j-1}$$

Therefore, it is enough to prove that

$$\pi_{\mathcal{S}^\alpha,v}(t){\restriction_v} = \pi_{\mathcal{W}_{i,j}^\alpha}\big(\pi_{\mathcal{S},v}(t){\restriction_v}\big). \tag{3}$$

Note that the only difference between the $(j-1)$-schemas $\mathcal{S}^\alpha$ and the j-schema $\mathcal{S}$ is the sets $R_v^{(j)} = E_v^\alpha$. That is, $\pi_{\mathcal{S}^\alpha,v}(t)$ operates as $\pi_{\mathcal{S},v}(t)$ but additionally every first occurrence w of j under v is replaced by $\top$ or $\bot$ depending whether $t \in E_w^\alpha$. By the inductive assumption (2), this is equivalent to checking whether the subtree $\pi_{\mathcal{S},v}(t){\restriction_w}$ belongs to $\mathcal{W}_{i,j}^\alpha$. Therefore, (3) follows. □

The proof of Proposition 3 is concluded by the following fact and the measurability of $\bigcup_{\alpha<\omega_1} E_v^\alpha$ that follows from (1).

Fact 2. *For every $v \in \{0,1\}^*$ we have $\bigcup_{\alpha<\omega_1} E^\alpha_v = \mathcal{W}_{\exists,v}(\mathcal{S})$.*

Proof. By (2) and the definition of $\mathcal{W}_{\exists,v}(\mathcal{S})$ we have

$$E^\alpha_v = \{t \mid \pi_{\mathcal{S},v}(t)\restriction_v \in \mathcal{W}^\alpha_{i,j}\}$$
$$\mathcal{W}_{\exists,v}(\mathcal{S}) = \{t \mid \pi_{\mathcal{S},v}(t)\restriction_v \in \mathcal{W}_{i,j}\}$$

Therefore, the fact follows directly from Theorem 5. □

3.4 Proof of Theorem 2

We conclude this section by proving Theorem 2 using the above construction.

Proof. If $j = k$ and $\mathcal{S}$ is the trivial j-schema $()_v$ then by (2) for every $v \in \{0,1\}^*$ and $\alpha < \omega_1$ we have

$$E^\alpha_v = \{t \mid t\restriction_v \in \mathcal{W}^\alpha_{i,k}\},$$

because the projection $\pi_{\mathcal{S},v}(t)$ in that case does not modify the given tree t. In particular, for $v = \epsilon$ we have $E^\alpha_\epsilon = \mathcal{W}^\alpha_{i,k}$.

Therefore, Theorem 2 follows from (1) for $v = \epsilon$ and the fact that U has μ-measure 0. □

4 Kolmogorov's $\mathcal{R}$–sets

In this section we provide the basic definitions and state the main results of the theory of $\mathcal{R}$-sets. The expository article of Kanovei [22] constitutes an excellent introduction to the topic.

As outlined in the introduction, one of Kolmogorov's motivations for investigating $\mathcal{R}$-sets was to identify a large σ-algebra of measurable (and more generally, "well behaved") sets. The approach followed by Kolmogorov [25] for obtaining such a σ-algebra is based on the identification of a family $\mathcal{F}$ of "safe" operations on sets guaranteeing that, for $f \in \mathcal{F}$, the set $f(X_1, \ldots, X_m, \ldots)$ is measurable whenever every set X_i of the input sequence (X_i) is a measurable set. Clearly, the operations of *countable* union ($\bigcup$), *countable* intersection ($\bigcap$) and complementation ($\neg$) are safe operations. Another important safe operation, today well-known as the *Suslin operation* ($\mathcal{A}$) had been discovered in 1917 (see, e.g. [24, §14.C]). Kolmogorov's insight was to generalize this idea and define a *transform* operator $\mathcal{R}$ mapping safe operation f to a new, more expressive safe operation $\mathcal{R}(f)$. As we will discuss below, it is the case that $\mathcal{R}(\bigcup) = \mathcal{A}$, and further iterations of $\mathcal{R}$ and complementation produce strictly more expressive operations. Using the $\mathcal{R}$ transform, Kolmogorov defined the σ-algebra of $\mathcal{R}$-sets as the least σ-algebra containing the open sets and closed under $\mathcal{F}$, where the family $\mathcal{F}$ is itself obtained by closing the familiar operations $\{\bigcup, \bigcap, \neg\}$ under the $\mathcal{R}$ transform. An equivalent definition, more convenient for our purposes, is obtained by considering the least σ-algebra containing the *clopen* sets and closed under the the family $\mathcal{F}$ obtained by closing the single operation $\bigcup \circ \bigcap$

(see Definition 3) under the co-$\mathcal{R}$ operation (see Definition 20). After this brief informal introduction, we now proceed with the formal definitions.

In the rest of this section we fix a zero-dimensional Polish space X. A (countable) *operation on sets* is a function $\Gamma\colon (powerset(X))^\omega \to \mathcal{P}(X)$. Note that, e.g., the operation of complementation can be seen as a countable operation ignoring all but its first input: $\neg(A_0,\dots,A_n,\dots) = X \setminus A_0$.

Among the family of all operations, an important subfamily is that of *positive analytic operations*. Informally, these are the operations that are monotone and such that the question "does x belong to $\Gamma(\{A_n\})$" is completely determined by the set of indices $\{n \mid x \in A_n\}$.

Definition 15. *A* positive analytic operations *is an operation Γ such that, for any two sequences $\{A_n\}$ and $\{B_n\}$, it holds that:*

1. *(monotonicity) $\forall n(A_n \subseteq B_n) \Rightarrow \Gamma(\{A_n\}) \subseteq \Gamma(\{B_n\})$, and*
2. *$\forall n(x \in A_n \Leftrightarrow y \in B_n) \Rightarrow \big(x \in \Gamma(\{A_n\}) \Leftrightarrow y \in \Gamma(\{B_n\})\big)$*

An alternative and very convenient description of positive analytic operations can be given by introducing the concept of *basis* of an operation Γ.

Definition 16. *We say that $N \subseteq \mathcal{P}(\omega)$ is a* basis *for the operation Γ if*

$$\Gamma(\{A_n\}) = \bigcup_{S\in N} \bigcap_{n\in S} A_n$$

For any $N \subseteq \mathcal{P}(\omega)$, the unique operation induced by N is denoted by Γ_N.

Note that the union is uncountable if N is uncountable. The following proposition is due to Kantorovich and Livenson ([23, Theorem 1, page 230])

Proposition 4. *An operation Γ is positive analytic if and only if there exists $N \subseteq \mathcal{P}(\omega)$ such that $\Gamma = \Gamma_N$.*

Remark 3. Observe that if N is a basis for Γ, then also its upward-closure $N' = \{X \in \mathcal{P}(\omega) \mid \exists Y \in N.(Y \subseteq X)\}$ is a basis for Γ. Hence, we can always assume that the basis N of a positive analytic operation is an upward closed set.

Example 1. The countable union operation ($\bigcup$) is positive analytic with, e.g., basis $N = \{\{n\} \mid n \in \omega\}$. Similarly, the operation of countable intersection has basis $N = \{\omega\}$.

In the rest of this article we only consider positive analytic operations, henceforth referred to simply as *operations*. It will be convenient, in what follows, to assume that the countably many arguments of an operation Γ are indexed by a countable set (called the *arena* of Γ) denoted by $\mathbb{A}_\Gamma$. Thus an operation Γ has type $\Gamma\colon \mathcal{P}(X)^{\mathbb{A}_\Gamma} \to \mathcal{P}(X)$. Clearly this is just a useful notational convention, since ω and $\mathbb{A}_\Gamma$ can be put in bijective correspondence. A basis for Γ is then now taken to be a collection $N \subseteq \mathcal{P}(\mathbb{A}_\Gamma)$ such that $\Gamma(A_s) = \bigcup_{S\in N} \bigcap_{s\in N} A_s$.

Example 2. The Souslin operation $\mathcal{A}$ is defined (see, e.g. [24, §14.C]) using $\mathbb{A}_{\mathcal{A}} = \omega^*$, the set of finite sequences (including the empty sequence ϵ) of natural numbers. The basis N of $\mathcal{A}$ is the set of maximal chains (under the prefix relation) of sequences. In other words, each $S \in N$ is the set of all prefixes of an *infinite* sequence of natural numbers. The Souslin operation is defined as:

$$\mathcal{A}(\{B_s\}_{s\in\mathbb{A}_{\mathcal{A}}}) = \bigcup_{S\in N} \bigcap_{s\in S} B_s$$

We now define *transforms* (operators) on operations, as outlined at the beginning of this section.

Definition 17 (Composition). *Given two operations Γ and Θ their composition $\Theta \circ \Gamma$ is the operation with arena $\mathbb{A}_\Gamma \times \mathbb{A}_\Theta$ defined as:*

$$\Theta \circ \Gamma(\{A_{s,s'} \mid s \in \mathbb{A}_\Gamma, s' \in \mathbb{A}_\Theta\}) = \Theta(\{\ \Gamma(\{A_{s,s'} \mid s \in \mathbb{A}_\Gamma\}) \mid s' \in \mathbb{A}_\Theta\})$$

A basis for $\Theta \circ \Gamma$ is can by given ([22, §1]) by $N \subseteq \mathcal{P}(\mathbb{A}_\Gamma \times \mathbb{A}_\Theta)$ consisting of all pairs $S \times S'$ of the form $\{\langle s, s'\rangle \mid s \in S \wedge s' \in S'\}$, with $S \in N_\Gamma$ and $S' \in N_\Theta$.

Example 3. The operation $\bigcup_n \bigcap_m A_{n,m}$ coincides with $\bigcup \circ \bigcap$.

Definition 18 (Dualization). *For a given operation Θ with arena $\mathbb{A}_\Theta$ and basis N_Θ, we define a* dual *operation* co-Θ *with the same arena $\mathbb{A}$, defined as:*

$$\text{co-}\Theta(\{A_s \mid s \in \mathbb{A}\}) = \bigcap_{S\in N_\Theta} \bigcup_{s\in S} A_s$$

A basis for co-Θ is given by $N_{\text{co-}\Theta} \stackrel{\text{def}}{=} \{S \in \mathcal{P}(\mathbb{A}) \mid \forall T \in N_\Theta.(T \cap S \neq \emptyset)\}$. See, e.g. [22, §1].

Example 4. The following equalities hold: co-$\bigcup = \bigcap$ and co-$\bigcap = \bigcup$.

Proof. Recall that $\mathbb{A}_\bigcup = \mathbb{A}_\bigcap = \omega$ with $N_\bigcup = \{\{n\} \mid n \in \omega$ and $N_\bigcap = \{\omega\}$. Then $N_{\text{co-}\bigcup}$ consists of sets which have nonempty intersection with every singleton. But there is only one such set: ω. Hence, $N_{\text{co-}\bigcup} = N_\bigcap$.

The second equality is a bit more involved. By unfolding the definitions we have

$$x \in \text{co-}\bigcap(\{A_n\}) \Leftrightarrow \exists S \subseteq \omega.(S \cap \omega \neq \emptyset) \wedge (\forall n \in S.\ x \in A_n)$$

In the right expression is satisfied, then it is satisfied by a minimal $S = \{n\}$, for some $n \in \omega$. Therefore the right condition is equivalent to $\exists n \in \omega.(x \in A_n)$, i.e., $x \in \bigcup(\{A_n\})$. □

Definition 19 ($\mathcal{R}$-transform). *The $\mathcal{R}$-transform of an operation Θ with basis N_Θ is the operation $\mathcal{R}\Theta$ with arena $\mathbb{A}_\mathcal{R} = (\mathbb{A}_\Theta)^*$ (finite sequences of elements in $\mathbb{A}_\Theta$) and basis:*

$$N_{\mathcal{R}\Theta} \stackrel{\text{def}}{=} \{S \subseteq (\mathbb{A}_\Theta)^* \mid \epsilon \in S \wedge \forall t \in S\ \{v \in \mathbb{A}_\Theta \mid tv \in S\} \in N_\Theta\} \qquad (4)$$

where ϵ denotes the empty sequence and tv the concatenation of $t \in (\mathbb{A}_\Theta)^*$ with $v \in \mathbb{A}_\Theta$.

The definition of the basis can be read as follows. The elements $S \in N_{\mathcal{R}\Theta}$ are sets of finite sequences which, when ordered by the prefix relation, can be seen as trees with root ϵ. Then a tree S is in $N_{\mathcal{R}\Theta}$ if and only if, for all of its vertices $t \in S$, the set $\{v \in \mathbb{A}_\Theta \mid tv \in S\}$ corresponding to the children of t in S, is in N_Θ. Hence $N_{\mathcal{R}\Theta}$ is a set of trees whose possible shapes are determined by N_Θ.

We now propose the following simple example to illustrate the definition of the $\mathcal{R}$ transform.

Example 5. The following equality holds: $\mathcal{R}\bigcup = \mathcal{A}$.

Proof. Recall that $\mathbb{A}_{\bigcup} = \omega$ and $N_{\bigcup} = \{\{n\} \mid n \in \omega\}$. By definition, $\mathbb{A}_{\mathcal{R}\bigcup}$ is ω^* which is indeed the arena of $\mathcal{A}$ as in Example 2. Hence, we just need to check that $N_{\mathcal{A}} = N_{\mathcal{R}\bigcup}$. This follows directly from definitions since an element $S \in N_{\mathcal{R}\bigcup}$ is a linearly order (by prefix relation) set of finite sequences, and it can be seen as an infinite tree with only one infinite branch. □

Definition 20. *We denote with co-$\mathcal{R}$ the composition of* co- *and $\mathcal{R}$ transforms and define operations*

$$\Theta_k \stackrel{\text{def}}{=} (co\text{-}\mathcal{R})^k \Big(\bigcup \circ \bigcap\Big).$$

where $\bigcup \circ \bigcap$ is as in Example 3.

Definition 21. *For a positive number $k \geq 1$, we say that a set $A \subseteq X$ is an $\mathcal{R}$-set of k-th level if and only if $A = \Theta_k(\{U_s \mid s \in \mathbb{A}_{\Theta_k}\})$ for some $\mathbb{A}_{\Theta_k}$-indexed family clopen sets $U_s \subseteq X$.*

In what follows, by $\mathcal{R}$-sets we mean $\mathcal{R}$-sets of finite levels. The rest of this subsection is devoted to the proofs of basic properties of $\mathcal{R}$–sets.

Lemma 5. *The k-th level of $\mathcal{R}$-sets is closed under pre-images of continuous functions.*

Proof. Continuous pre-images of clopen sets are clopen. Therefore the following equation, valid for an arbitrary operation Θ, concludes the proof

$$\begin{aligned} f^{-1}(\Theta(\{E_s\})) &= f^{-1}\Big(\bigcup_{N \in N_\Theta} \bigcap_{s \in N} E_s\Big) = \\ &= \bigcup_{N \in N_\Theta} \bigcap_{s \in N} f^{-1}(E_s) = \\ &= \Theta\big(\{f^{-1}(E_s)\}\big). \end{aligned} \tag{5}$$

□

Proposition 5 (Normality of $\mathcal{R}$). *For a given operator Θ the classes of sets which can be obtained by operators $\mathcal{R}\Theta$ and $\mathcal{R}\Theta \circ \mathcal{R}\Theta$ are the same.*

This theorem is proved in Kolmogorov's seminal work [25]. The proof is essentially a direct generalization of the analogous result about normality of the Souslin operation (see, e.g. [24, Proposition 25.6]). See also [16], [22, page 130] and [4, Lemma 2.17(i), page 17].

We say that an operation Γ *preserves measurability* if for any family $\mathcal{E} = \{E_s\}_{s \in \mathbb{A}_\Gamma}$ of measurable sets, the set $\Gamma(\mathcal{E})$ is measurable. The following theorem motivates our interest on the notion of $\mathcal{R}$-sets.

Theorem 6 (Kolmogorov). *If Γ and Θ preserve measurability then $\Gamma \circ \Theta$, $\mathcal{R}\Gamma$, and* co-Γ *preserve measurability.*

The theorem was proved by Kolmogorov [35, Theorem 6 in Appendix 2, page 273]. The proof circulated in the form of a manuscript and for the first time was published in a paper of Lyapunov [29, Theorem 4]. See also Section 7 in [22].

Corollary 1. *All $\mathcal{R}$-sets are measurable.*

5 Matryoshka games

In this section we define *Matryoshka games*, a variant of parity games which makes it easier to establish a connection with the operations Θ_k (see Definition 20 in Section 4) and thus with the finite levels of the hierarchy of $\mathcal{R}$-sets.

A Matryoshka game is the familiar structure of a two-player parity game played on an infinite countably branching graph, extended with a *labelling function* assigning to each finite play (i.e. every sequence of game-states ending in a terminal state) a *play label.* Formally:

$$\mathcal{G} = \{V^{\mathcal{G}} = V^{\mathcal{G}}_{\exists} \sqcup V^{\mathcal{G}}_{\forall}, F^{\mathcal{G}}, E^{\mathcal{G}}, v^{\mathcal{G}}_{I}, \Omega^{\mathcal{G}}, \mathbb{A}^{\mathcal{G}}, \mathrm{label}^{\mathcal{G}}\},$$

such that $\{V^{\mathcal{G}} = V^{\mathcal{G}}_{\exists} \sqcup V^{\mathcal{G}}_{\forall}, F^{\mathcal{G}}, E^{\mathcal{G}}, v^{\mathcal{G}}_{I}, \Omega^{\mathcal{G}}\}$ is a standard parity game with terminal positions $F^{\mathcal{G}}$. Let us define precisely all the elements:

- $V^{\mathcal{G}}$ is a countable set of *positions* of the game,
- $F^{\mathcal{G}}$ is a countable set of *terminal positions* of the game,
- $E^{\mathcal{G}} \subseteq V^{\mathcal{G}} \times \left(V^{\mathcal{G}} \cup F^{\mathcal{G}}\right)$ is the edge relation,
- $v^{\mathcal{G}}_{I} \in V^{\mathcal{G}}$ is the *initial position*,
- $\Omega^{\mathcal{G}}\colon V^{\mathcal{G}} \to \{i, \ldots, k\} \subseteq \omega$ is the *priority* function,
- $\mathbb{A}^{\mathcal{G}}$ is a set of *play labels*,
- $\mathrm{label}^{\mathcal{G}}\colon (V^{\mathcal{G}})^{*}F^{\mathcal{G}} \to \mathbb{A}^{\mathcal{G}}$ is a function assigning to finite plays their *play labels.*

Additionally, $\mathbb{A}^{\mathcal{G}}$ is a set of *play labels*, and $\mathrm{label}^{\mathcal{G}}\colon (V^{\mathcal{G}})^{*}F^{\mathcal{G}} \to \mathbb{A}^{\mathcal{G}}$ is a function assigning to finite plays their *play labels.*

We assume that for every $v \in V^{\mathcal{G}}$ there is at least one $v' \in V^{\mathcal{G}} \cup F^{\mathcal{G}}$ such that $(v, v') \in E^{\mathcal{G}}$, so that the only terminal game-states are in $F^{\mathcal{G}}$. As for standard parity games, the pair (i, k) containing the minimal and maximal values of Ω is called the *index* of the game. By $P \in \{\exists, \forall\}$ we denote the *players* of the game. The opponent of P is denoted by $\bar{P}$.

A *play* is defined as usual as a maximal path in the arena, i.e. either as a finite sequence in $\left(V^{\mathcal{G}}\right)^{*}F^{\mathcal{G}}$ or as an infinite sequence $(V^{\mathcal{G}})^{\omega}$. Similarly, a strategy σ for a player P is a function $\sigma\colon \left(V^{\mathcal{G}}\right)^{*}V^{\mathcal{G}}_{P} \to V^{\mathcal{G}} \cup F^{\mathcal{G}}$ defined as expected.

The novelty in Matryoshka games is given by the set of play labels $\mathbb{A}^{\mathcal{G}}$ and the associated labelling function $\mathrm{label}^{\mathcal{G}}$. These are used to define *parametric* winning condition in the Matryoshka game, as we now describe.

A set of play labels $X \subseteq \mathbb{A}^{\mathcal{G}}$ is called a *promise*. A finite play π is *winning for* $\exists$ *with promise* X if $\text{label}(\pi) \in X$. An infinite play is π is winning for $\exists$ if $\left(\limsup_{n\to\infty} \Omega^{\mathcal{G}}(\pi(n))\right)$ is even, as usual. If a play is not winning for $\exists$ then it is winning for $\forall$. A strategy σ for Player P is *winning in the Matryoshka game* $\mathcal{G}$ *with promise* X if, for every counter-strategy τ of $\overline{P}$, the resulting play $\pi(\sigma, \tau)$ is winning for P with promise X, in the sense just described. The following proposition directly follows from the well-known determinacy of parity games.

Proposition 6. *If* $\mathcal{G}$ *is a Matryoshka game with play labels* $\mathbb{A}^{\mathcal{G}}$ *and* $X \subseteq \mathbb{A}^{\mathcal{G}}$ *then exactly one of the players has a winning strategy in* $\mathcal{G}$ *with promise* X.

Proof. By reduction to the standard parity games: first, we can assume that we play on the unravelling of the arena with additional loop-edges on elements of $F^{\mathcal{G}}$. For a given promise $X \subseteq \mathbb{A}^{\mathcal{G}}$ we can set the priorities on $F^{\mathcal{G}}$ such that the position in the unravelling corresponding to $f \in F^{\mathcal{G}}$ is winning for $\exists$ if and only if the label of the unique play reaching this position belongs to X.

A winning strategy for P in the obtained game gives a X-winning strategy for P in the original game. □

The point of having parametrized winning conditions in Matryoshka games is the possibility of defining set-theoretical operations with a direct game interpretation. Given a Polish space X, the *operation* on sets (see Section 2) associated with a Matryoshka game $\mathcal{G}$ has arena $\mathbb{A}^{\mathcal{G}}$ and is defined as follows:

$$\mathcal{G}(\mathcal{E}) \stackrel{\text{def}}{=} \left\{x \in X \;\mid \exists \text{ has a w. s. in } \mathcal{G} \text{ with promise } \{s \in \mathbb{A}^{\mathcal{G}} \mid x \in E_s\}\right\} \qquad (6)$$

where $\mathcal{E} = \{E_s \mid s \in \mathbb{A}^{\mathcal{G}}\}$ is a family of subsets of X.

We now define a Matryoshka game called $\mathcal{G}_0$, whose associated operation is precisely the operation $(\bigcup \circ \bigcap)$ (as in Example 3 of Section 2). The structure of $\mathcal{G}_0$ is depicted in Figure 2. This is a simple game of two steps, where $\exists$ chooses a number n and $\forall$ responds choosing a number m.

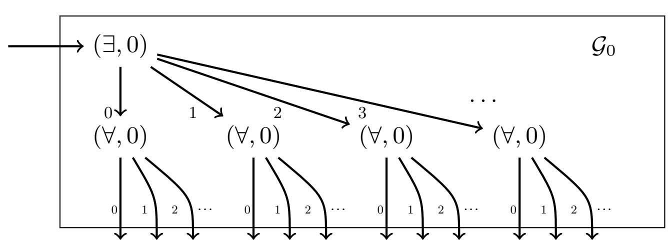

Fig. 2. The game $\mathcal{G}_0$ corresponding to the operation $\bigcup \circ \bigcap$.

More formally, let the arena $V^{\mathcal{G}_0}$ consist of sets of positions $V_{\exists}^{\mathcal{G}_0} = \{e_0\}$ and $V_{\forall}^{\mathcal{G}_0} = \{a_0, a_1, \ldots\}$ and let $F^{\mathcal{G}_0} = \{f_{n,m} \mid n, m \in \mathbb{N}\}$. Let $E^{\mathcal{G}_0}$ contain pairs of the form (e_0, a_n) and $(a_n, f_{n,m})$ for $n, m \in \mathbb{N}$. Let $\mathbb{A}^{\mathcal{G}_0} = \omega^2$. Note that all the plays of $\mathcal{G}_0$ are finite and of the form $\pi = (e_0, a_n, f_{n,m})$. For such a play, let $\text{label}^{\mathcal{G}_0}(\pi) = (n, m)$. Let $\Omega^{\mathcal{G}_0} \colon V^{\mathcal{G}_0} \to \{0\}$ be the constant function.

We now introduce *transformations* on games which directly match the corresponding transformations on operations defined in Section 2.

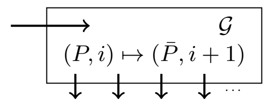

Fig. 3. The game co-$\mathcal{G}$.

Game co-$\mathcal{G}$*.* For a Matryoshka game $\mathcal{G}$ of index (i, k), we define co-$\mathcal{G}$ as the game obtained from $\mathcal{G}$ by replacing the sets $V_\exists \leftrightarrow V_\forall$ and increasing all priorities in Ω by 1. Note that the index of co-$\mathcal{G}$ is $(i + 1, k + 1)$, and that the sets of plays in the two games are equal. We define $\mathbb{A}^{\text{co-}\mathcal{G}} \stackrel{\text{def}}{=} \mathbb{A}^{G}$ and $\text{label}^{\text{co-}\mathcal{G}}(\pi) \stackrel{\text{def}}{=} \text{label}^{\mathcal{G}}(\pi)$.

Game $\mathcal{RG}$*.* Lastly, we define the $\mathcal{R}$ transformation on games. Let us take a Matryoshka game $\mathcal{G}$ of index (i, k). Let $2j$ be the minimal even number such that $k \leq 2j$. The game $\mathcal{RG}$ is depicted in Figure 4.

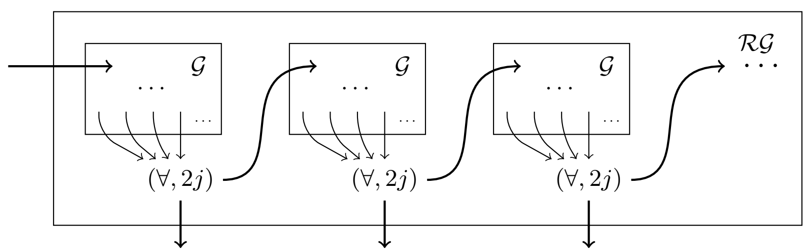

Fig. 4. The game $\mathcal{RG}$.

A play in the game $\mathcal{RG}$ starts from a first copy of $\mathcal{G}$. In this *inner* game, the play π can either be infinite (in which case π is a valid play in $\mathcal{RG}$ and is winning for Player P if and only if it is winning for P in $\mathcal{G}$) or terminate in a terminal state of $\mathcal{G}$. In this latter case, Player $\forall$ can either conclude the game $\mathcal{RG}$, or start another session of the inner game $\mathcal{G}$. Observe that if $\forall$ always chooses to start a new session, he loses because the even priority $2j$ is maximal in $\mathcal{RG}$.

The set of play labels $\mathbb{A}^{\mathcal{RG}}$ is defined as $\left(\mathbb{A}^{\mathcal{G}}\right)^*$, i.e. the set of finite sequences of play labels in $\mathcal{G}$. The set of positions $V^{\mathcal{RG}}$ of $\mathcal{RG}$ is defined as $\{a_0, a_1, \ldots\} \sqcup \omega \times V^{\mathcal{G}}$. Each vertex a_n belongs to $\forall$ (i.e. $a_n \in V_\forall^{\mathcal{RG}}$). A vertex $(n, v) \in \omega \times V^{\mathcal{G}}$ belongs to a player P if and only if $v \in V_P^{\mathcal{G}}$. $\mathcal{RG}$ has infinitely many terminal positions $f_0, f_1, \ldots$ The priority function on $\omega \times V^{\mathcal{G}}$ is the same as in $\mathcal{G}$. All the vertices a_n have priority $2j$. The edges in $\mathcal{RG}$ are of the following forms:

– if $(v, v') \in E^{\mathcal{G}}$ with $v, v' \in V^{\mathcal{G}}$ then $((n, v), (n, v')) \in E^{\mathcal{RG}}$ for $n \in \mathbb{N}$,
– if $(v, f) \in E^{\mathcal{G}}$ with $v \in V^{\mathcal{G}}$ and $f \in F^{\mathcal{G}}$ then $((n, v), a_n) \in E^{\mathcal{RG}}$ — instead of a terminal position of $\mathcal{G}$ we move to the successive vertex of $\forall$,
– additionally, we add edges $(a_n, (n + 1, v_I^{\mathcal{G}})) \in E^{\mathcal{RG}}$ and $(a_n, f_n) \in E^{\mathcal{RG}}$.

Let the initial position of $\mathcal{RG}$ be $(0, v_I^{\mathcal{G}})$.

The crucial part of the definition of the transformation $\mathcal{R}$ are the labels. Consider a finite play π that reaches a terminal position f_n of $\mathcal{RG}$. Such a play has lasted for n rounds until it reached the terminal position f_n. In that case, the play π is of the form:

$$a_0\pi_0 a_1\pi_1 \ldots \pi_{n-1} a_n f_n$$

where π_i corresponds to a play in $\mathcal{G}$. Let x_i be the label assigned by $\mathcal{G}$ to the play π_i and let

$$\mathrm{label}^{\mathcal{RG}}(\pi) = \big(x_0, x_1, \ldots, x_{n-1}\big).$$

Given the basic Matryoshka game G_0 and the two transformations of games co- and $\mathcal{R}$, we can construct more and more complex "nested" games. This fact motivates the name *Matryoshka* for this class of games. We denote with $\mathcal{G}_k$ the game obtained from G_0 by iterating k-times the composed transformation (co-$\mathcal{R}$).

By the definition, the game $\mathcal{G}_k$ for $k > 0$ consists of infinitely many copies of $\mathcal{G}_{k-1}$ and an additional set of new vertices as depicted on Figure 4. These new vertices are called the *k-layers* of the game. Therefore, by unfolding the definition, each vertex v of $\mathcal{G}_k$ is either a vertex of a copy of $\mathcal{G}_0$ or it belongs to a j-layer for some $1 \leq j \leq k$. Observe that if v is in a j-layer of $\mathcal{G}_k$ then

$$\Omega^{\mathcal{G}_k}(v) = k{+}j{-}1 \quad \text{and} \quad \Big(v \in V_\forall^{\mathcal{G}_k} \Leftrightarrow k{+}j{-}1 \equiv 0 \pmod 2\Big). \tag{7}$$

We are now ready to state the expected correspondence between the operation Θ_k of Section 2 and the Matryoshka game $\mathcal{G}_k$.

Theorem 7. *For every $k \in \omega$ the basis N_{Θ_k} of the Θ_k operation equals the family* $\mathrm{promise}(\mathcal{G}_k) \stackrel{\text{def}}{=} \{X \subseteq \mathbb{A}_k \mid \exists$ *has a winning strategy in $\mathcal{G}_k$ with promise X*$\}$.

Proof. The proof goes by induction. First take $k = 0$. Note that the following family forms a basis of $\Theta_0 = \bigcup \circ \bigcap$:

$$N_{\Theta_0} \;=\; \Big\{N \subseteq \omega^2 \mid \exists_n \forall_m \; (n, m) \in N\Big\}.$$

Observe that a strategy of $\exists$ in $\mathcal{G}_0$ coincides with the selection of the first number n. Then $\forall$ selects the second number m and the play ends in a terminal position with label (n, m). Therefore, the family of promises of winning strategies of $\exists$ in $\mathcal{G}_0$ coincides with N_{Θ_0}.

Now assume that $N_\Theta = \text{promise}(\mathcal{G})$, we prove that $N_{\text{co-}\Theta} = \text{promise}(\text{co-}\mathcal{G})$. Let $\mathbb{A}$ be the play labels in $\mathcal{G}$. Observe that the following conditions are equivalent (by w.s. we abbreviate winning strategy):

$$
\begin{aligned}
X &\in N_{\text{co-}\Theta} \\
&\qquad \text{by the definition of co-}\Theta \\
\forall_{X' \in N_\Theta}\ X \cap X' &\neq \emptyset \\
&\qquad N_\Theta \text{ is upward-closed (Remark 3)} \\
\mathbb{A} \setminus X &\notin N_\Theta \\
&\qquad \text{by the inductive assumption} \\
\text{there is no } (\mathbb{A}\setminus X)\text{-w. s. for } \exists \text{ in } &\mathcal{G} \\
&\qquad \text{by the definition of co-}\mathcal{G} \\
\text{there is no } X\text{-w. s. for } \forall \text{ in co-}&\mathcal{G} \\
&\qquad \text{by determinacy (Proposition 6)} \\
\exists \text{ has a } X\text{-w. s. in co-}&\mathcal{G} \\
&\qquad \text{by the definition of promise(co-}\mathcal{G}) \\
X &\in \text{promise}(\text{co-}\mathcal{G})
\end{aligned}
$$

Now assume that $N_\Theta = \text{promise}(\mathcal{G})$, we prove that $N_{\mathcal{R}\Theta} = \text{promise}(\mathcal{R}\mathcal{G})$. This will finish the inductive proof of the proposition. As above, let $\mathbb{A}$ equal arena($\mathcal{G}$). Additionally, let $\mathcal{G}^i$ denote the sub-game of $\mathcal{R}\mathcal{G}$ corresponding to the i-th copy of $\mathcal{G}$ (formally, $\mathcal{G}^i$ contains vertices of the form (i, v)).

First assume that σ is a X-winning strategy for $\exists$ in $\mathcal{R}\mathcal{G}$. We need to show that $X \in N_{\mathcal{R}\mathcal{G}}$. Clearly $\epsilon \in X$ since $\forall$ can move directly from a_0 to e_0. Let $\bar{s} \in X$. We need to show that $\{x \mid \bar{s}x \in X\}$ is an element of $N_{\mathcal{G}}$. Let $i = |\bar{s}|$ be the length of s. Observe that $\bar{s} \in X$ means that there exists a finite play π that is consistent with σ that goes through the sub-games $\mathcal{G}^0, \ldots, \mathcal{G}^{i-1}$ and then to a_i and e_i, formally

$$\pi = a_0 \pi_0 a_1 \pi_1 \cdots a_{i-1} \pi_{i-1} a_i e_i.$$

Consider the strategy of $\exists$ σ' in $\mathcal{G}$ obtained as restricting σ to sequences that extend $a_0 \pi_1 \cdots \pi_{i-1} a_i (i, v_I^{\mathcal{G}})$, where $(i, v_I^{\mathcal{G}})$ is the initial position in the i-copy of $\mathcal{G}$. This strategy is winning with some minimal guarantee $X' \subseteq \mathbb{A}$. Note that if there is a play π' consistent with σ' such that $\text{label}^{\mathcal{G}}(\pi') = x$ then $\bar{s}x \in X$ — directly after the play π', player $\forall$ can decide to move from a_{i+1} to e_{i+1}. Therefore, σ' witnesses that $\{x \mid \bar{s}x \in X\} \in N_{\mathcal{G}}$.

Now assume that $X \in N_{\mathcal{R}\Theta}$. In particular $X \subseteq \mathbb{A}^*$ and for every element $\bar{s} \in X$ we have $\{x \mid \bar{s}x \in X\} \in N_{\mathcal{G}}$. We need to construct a X-winning strategy σ of $\exists$ in $\mathcal{R}\mathcal{G}$. The strategy is defined inductively, between successive sub-games $\mathcal{G}^i$. The invariant says, that if a play π consistent with σ reaches the node e_i then $\text{label}^{\mathcal{R}\mathcal{G}}(\pi) \in X$. Assume that we have reached a_i after a play π such that the label of πe_i is $\bar{s}$. By the invariant, we know that $\bar{s} \in X$. In particular, there exists a winning strategy σ' of $\exists$ in $\mathcal{G}$ with the guarantee $\{x \mid \bar{s}x \in X\}$. Let σ follow the decisions of σ' until reaching a terminal position of $\mathcal{G}$ (i.e. the position

a_{i+1} in $\mathcal{RG}$). We now prove that σ is X-winning. Let π be a play consistent with σ. There are the following cases:

- π is a finite play and by the above invariant $\text{label}^{\mathcal{RG}}(\pi) \in X$.
- π is an infinite play that stays from some point on in one of the sub-games $\mathcal{G}^i$. In that case π is winning for $\exists$ since it contains a winning play in $\mathcal{G}$ as a suffix.
- π is an infinite play that passes through infinitely many sub-games $\mathcal{G}^i$. In that case all the vertices a_i are on π so

$$\limsup_{n\to\infty} \Omega(\pi(n)) = 2n$$

and therefore π satisfies the parity condition.

□

Corollary 2. *For each k and $(E_s)_{s\in\mathbb{A}_k}$ we have $\Theta_k\big((E_s)_{s\in\mathbb{A}_k}\big) = \mathcal{G}_k\big((E_s)_{s\in\mathbb{A}_k}\big)$.*

5.1 Relation between $\mathcal{R}$–sets and the index hierarchy

In this subsection we shall establish a precise correspondence between the finite levels of the hierarchy of $\mathcal{R}$-sets and game tree languages $\mathcal{W}_{i,k}$. The proofs will make use of the "intermediary" concept of Matryoshka games, introduced in the previous section.

As a preliminary step, it is convenient to define a variant of game tree languages defined on countably branching trees. This will simplify the connection with Matryoshka games which are played on countably branching structures.

Definition 22. *Let $\mathrm{Tr}^{\omega}_{i,k}$ be the space of labelled ω-trees*

$$t\colon \omega^* \to \{\exists,\forall\} \times \{i,\ldots,k,\top,\bot\}.$$

Each $t \in \mathrm{Tr}^{\omega}_{i,k}$ is naturally interpreted as a parity game on the countable tree structure, with the possibility of terminating at leaves, labelled by $\top$ and $\bot$, which are winning for $\exists$ and $\forall$, respectively. We also require that the following simple technical conditions are satisfied:

1. *in the root there is a vertex (P,k) where $P=\exists$ if $i=0$ and $P=\forall$ if $i=1$,*
2. *for a vertex $v\in\omega^{2n}$ on an even depth in the tree, if the label of v is of the form (P,j) then the labels of all the vertices $vl\in\omega^*$ for $l\in\omega$ are $(\bar{P},j)$, see the picture below. In other terms, the players appear alternately and the priorities are duplicated every second level.*

Definition 23. *$\mathcal{W}^{\omega}_{i,k} \subseteq \mathrm{Tr}^{\omega}_{i,k}$ is the set of ω-trees such that $\exists$ has a winning strategy.*

An easy argument shows that

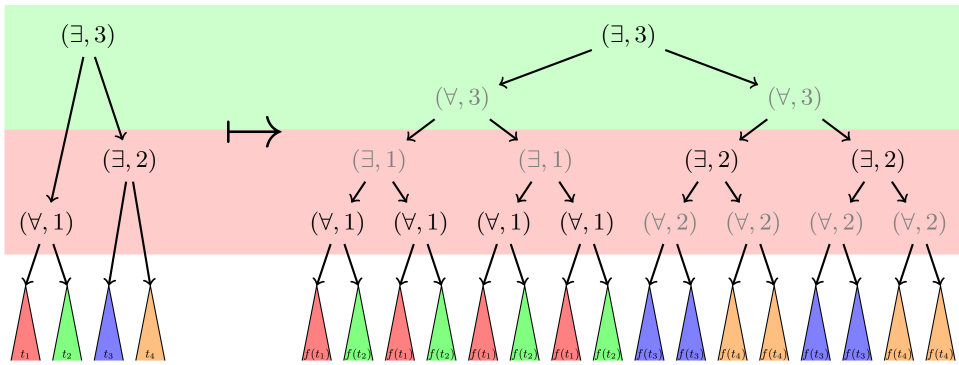

Fig. 5. "Normalization" of game languages — the same technique works for both binary and ω-branching trees.

Lemma 6. *Dropping conditions 1. and 2. in Definition 22 defines a language which is Wadge equivalent to* $\mathcal{W}^{\omega}_{i,k}$.

Proof. Let us start from an observation, that adding technical requirements regarding the same ranks on two subsequent levels $2n$ and $2n+1$ of a given tree and requiring that player $\exists$ and $\forall$ move one after another in turns does not limit generality from the point of view of Wadge reducibility. Namely, as illustrated by Figure 5 we may modify arbitrary graph to fulfil the additional requirements. □

The following routine lemma establishes the connection between ω-branching game tree languages $\mathcal{W}^{\omega}_{i,k}$ and binary (as in Section 2) game tree languages $\mathcal{W}_{i,k}$.

Lemma 7. *For* $i < k$ *the language* $\mathcal{W}_{i,k}$ *is Wadge equivalent to* $\mathcal{W}^{\omega}_{i+1,k}$. *In particular* $\mathcal{W}_{0,1} \sim_W \mathcal{W}^{\omega}_{1,1}$ *and* $\mathcal{W}_{1,3} \sim_W \mathcal{W}^{\omega}_{0,1}$.

Proof. Let start with a reduction of $\mathcal{W}^{\omega}_{i+1,k}$ to $\mathcal{W}_{i,k}$. To encode infinite branching we use a standard trick - each leftmost branch B in the binary tree is treated as a one vertex V. Right children of vertices in B are treated as children of V. To guarantee that a player P who can choose a child of V will always exit branch B, we label vertices along B with the lowest possible priority loosing for P (i.e. i or $i+1$). One should notice, that such labelling does not increase $\limsup$ of a play.

The Wadge reduction of the language $\mathcal{W}^{\omega}_{1,3}$ to $\mathcal{W}_{0,3}$ is shown in Figure 6 below.

Technically more involved is a reduction of $\mathcal{W}_{i,k}$ to $\mathcal{W}^{\omega}_{i+1,k}$. The proof below is an adaptation of the proof of Lemmas IV.5 and IV.6 in [12]. Without loss of generality let $i = 0$ (i.e. priority winning for $\exists$). A continuous reduction ϕ maps a tree $t \in \mathrm{Tr}_{i,k}$ into $\phi(t) \in \mathrm{Tr}^{\omega}_{i+1,k}$ and is defined as follows. When we encounter a vertex with priority greater than 0 it is copied without any change — we can

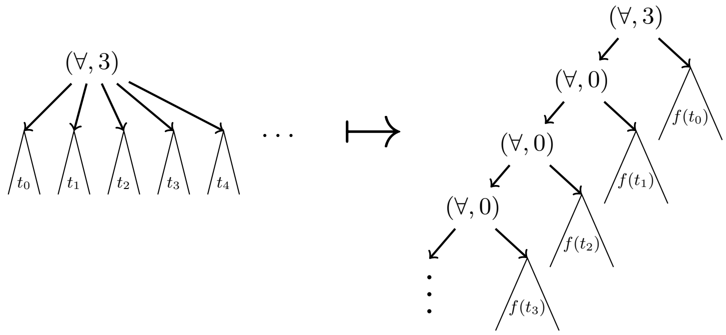

Fig. 6. Reduction of $\mathcal{W}^{\omega}_{1,3}$ to $\mathcal{W}_{0,3}$.

duplicate both of the children of this vertex infinitely many times, to make sure that the obtained tree is ω-branching.

The situation is different when we encounter a vertex v with priority 0. In this case vertex $v' = (\forall, 1)$ with ω children is produced. Intuitively, since priority 0 is loosing for $\forall$ he wants to visit vertices with higher priorities. Let τ_n $(n \in \omega)$ be a list of all strategies of $\forall$ starting from v. The successive children of v' correspond to the strategies $(\tau_n)_{n\in\mathbb{N}}$. In order to decide children of τ_n, we consider possible choices of $\exists$ against strategy τ_n. This gives finitely many options which we verbatim copy as children of τ_n, unless a priority of such child is 0. Then we decide that $\forall$ looses and mark it as $\top$. We will prove that ϕ is a Wadge reduction by showing the following equivalence

$$t \in \mathcal{W}_{i,k} \quad \text{if and only if} \quad \phi(t) \in \mathcal{W}^{\omega}_{i+1,k}.$$

The proof is based on the heuristic that if $\forall$ cannot reach a priority greater than 0 then he looses.

Assume first that σ is a winning strategy of $\exists$ on the tree t. We need to show that $\exists$ wins on $\phi(t)$. We play according to σ on $\phi(t)$ until there appears a vertex with priority 0. Assume now that $\forall$ selected a strategy τ_n. Since σ and τ_n define a unique answer of $\exists$ in t we can select the counterpart of this answer in the tree $\phi(t)$. Since σ is winning, the above strategy either reaches $\top$ or the parity condition is satisfied.

Assume now that σ is a winning strategy of $\exists$ on $\phi(t)$ and towards a contradiction assume that τ is a winning strategy of $\forall$ on t. We play these two strategies against each other as far as the priority 0 is not reached in t. If 0 is reached, then against σ we play a finite approximation τ_n of τ which avoids vertices of rank 0 (if every approximation of τ contains a 0-labelled leaf then according to the König's lemma we would be able to construct a path in τ containing only 0-labelled vertices). If σ selects a leaf w of τ_n in $\phi(t)$, we mimic the same gameplay in t. As a result the visited priorities in t and $\phi(t)$ must be the same, but this contradicts our assumption that the strategies σ and τ are winning for $\exists$ and $\forall$ respectively. $\square$

The fact that $\mathcal{W}_{i,k}$ corresponds to $\mathcal{W}^{\omega}_{i+1,k}$ reflects the cost of the translation of ω-branching games into binary games: an extra priority is required to mimic countably many choices by iterating binary choices. Thanks to this lemma, in Theorem 3 one can replace the languages $\mathcal{W}_{k-1,2k-1}$ with the languages $\mathcal{W}^{\omega}_{k,2k-1}$.

Having established this convenient correspondence between $\mathcal{W}_{k-1,2k-1}$ and the languages $\mathcal{W}^{\omega}_{k,2k-1}$, we can go back to our original intent.

The result of Corollary 2 can be read as follows. Every $\mathcal{R}$-set belonging to the finite levels of the $\mathcal{R}$-hierarchy is also definable by Matryoshka games. This allows us to state a first relationship between $\mathcal{R}$-sets and the hierarchy of game tree languages.

Lemma 8. *Given a Polish space X, let $A \subseteq X$ be a set defined by a Matryoshka game $A = \mathcal{G}_k\big((E_s)_{s\in\mathbb{A}_k}\big)$, for $k \geq 1$. Then $A \leq_W \mathcal{W}^{\omega}_{k,2k-1}$.*

Proof. By definition (see Equation 6) of membership in A, we have that $x \in A$ if and only if the game $\mathcal{G}_k$ with promise $\{s \in \mathbb{A}^{\mathcal{G}}_k \mid x \in E_s\}$ is winning for player $\exists$. Observe that $\mathcal{G}_k$ uses k-priorities and it's greatest priority is odd. Thus, let us assume without loss of generality, that $\mathcal{G}_k$ uses priority $(k, 2k-1)$. We define the function $f\colon X \to \mathrm{Tr}^{\omega}_{k,2k-1}$ as mapping x to the corresponding unfolded parity game, where each final positions s is replaced by $\top$ if $x \in E_s$ and $\bot$ if $x \notin E_s$, as described in the proof of Proposition 6. Since all sets E_s are clopen (and thus the corresponding characteristic function χ_{E_s} is continuous) the function f is continuous. Therefore we have that $x \in A \Leftrightarrow f(x) \in \mathcal{W}^{\omega}_{k,2k-1}$ and this concludes the proof. □

Corollary 3. *For every $\mathcal{R}$-set A belonging to the k-th level of the $\mathcal{R}$-hierarchy, it holds that $A \leq_W \mathcal{W}_{k-1,2k-1}$.*

In other words, the game tree language $\mathcal{W}_{k-1,2k-1}$ is *hard* for the sets belonging to the k-th level of the $\mathcal{R}$-hierarchy. We will now strengthen this result by showing that $\mathcal{W}_{k-1,2k-1}$ is *complete* for the k-th level of the $\mathcal{R}$-hierarchy. To do this we show that $\mathcal{W}^{\omega}_{k,2k-1}$ belongs to the k-level of the hierarchy of $\mathcal{R}$-sets.

We will do so by explicitly constructing a family $\mathcal{E}_k = \{E_s \mid s \in \mathbb{A}_k\}$ of clopen sets in $\mathrm{Tr}^{\omega}_{k,2k-1}$ such that $\Theta_k(\mathcal{E}_k) = \mathcal{W}^{\omega}_{k,2k-1}$, where $\mathbb{A}_k$ is the arena of the operation Θ_k. The construction requires some effort. First we recall, from Section 4 that the arena of the operation $\bigcup \circ \bigcap$ is $\mathbb{A}_0 = \{\langle n, m\rangle \mid n, m \in \omega\}$ (the pairs of natural numbers) and from the definition of the transformation $\mathcal{R}$ we have $\mathbb{A}_k = \big(\mathbb{A}_{k-1}\big)^*$. Thus, for all $k \in \omega$, $\mathbb{A}_k$ is a set of nested sequences of pairs of natural numbers. For a sequence $s \in \mathbb{A}_k$ we define the maps `flatten` and `prioritiesMap` such that `flatten`$(s) \in \mathbb{A}_0^{\star}$ and `prioritiesMap`$(s) \in \omega^*$.

The formal definitions of `flatten` and `prioritiesMap` using a natural Haskell data structure[6]. This is a `NestedList`, an abstraction of a list which naturally allows to consider $\mathbb{A}_k$, that is sequences of sequences of. . . of sequences.

[6] Code can be run locally on a computer or on-line on `fpcomplete` server; the service allows on-line modifications, in particular playing with more examples at the webpage `http://www.fpcomplete.com/user/henryk/kolmogorovflatmaps`.

```
-- run at http://www.fpcomplete.com/user/henryk/kolmogorovflatmaps
data NestedList a = Elem a | List [NestedList a]
                  deriving (Show)
-- straightforwardly define flatten
flatten :: NestedList a -> [a]
flatten (Elem x) = [x]
flatten (List x) = concatMap flatten x
-- prioritiesMap defines through auxialliary prioritiesMap'
prioritiesMap (x) = prioritiesMap'(x,0)

prioritiesMap' :: (NestedList a, Int) -> [Int]
prioritiesMap' (Elem a,n) = [n]
prioritiesMap' (List (x:[]), n) = prioritiesMap' (x,n+1)
prioritiesMap' (List (x:y:xs),n) = prioritiesMap' (x,0) ++
    prioritiesMap' (List( y:xs),n)
prioritiesMap' (List [],n)  = []
```

The map `flatten` takes a nested sequence in $\mathbb{A}_k$ and returns the "flattened" sequence, that is all the braces are removed, for example

$$\texttt{flatten}\big((((\langle 2,15\rangle)),((\langle 7,5\rangle),(\langle 6,4\rangle))))\big) = (\langle 2,15\rangle,\langle 7,5\rangle,\langle 6,4\rangle).$$

The function `prioritiesMap` computes the number of the closing brackets after each pair of natural numbers:

$$\texttt{prioritiesMap}\Big((((\langle 2,15\rangle)),((\langle 7,5\rangle),(\langle 6,4\rangle))))\Big) = (2,1,3).$$

We also define $\texttt{treeMap}(t,s)$ where $t \in \mathrm{Tr}^{\omega}_{k,2k-1}$ and $s \in \mathbb{A}_k$. Since we limited our attention to alternating trees, each vertex in the ω-branching tree t can be identified with a sequence of pairs of natural numbers. Then, if $s \in \mathbb{A}_k$, the function $\texttt{treeMap}(t,s)$ computes first $\texttt{flatten}(s)$ and returns the sequence of priorities assigned to the vertices along the path of t indicated by $\texttt{flatten}(s)$. On Figure 7 we have an example of a tree t where

$$\texttt{treeMap}\big(t,(((\langle 2,15\rangle)),((\langle 7,5\rangle),(\langle 6,4\rangle))))\big) = (2,1,3).$$

Define $\mathcal{E}_k = \{E_s \mid s \in \mathbb{A}_k\}$ such that for $t \in \mathrm{Tr}^{\omega}_{k,2k-1}$ we have $t \in E_s$ iff for

- $v = \texttt{prioritiesMap}(s)$,
- $b = \texttt{treeMap}(t,s)$,
- $L = \min\{k \in \omega \mid v(k) \neq b(k)\}$

$v \neq b$ holds, and either $b(L) = \top$ or

$$\min(b(L),v(L)) \equiv 0 \pmod 2. \tag{8}$$

It is simple to verify that the sets E_s are indeed clopen in the space $\mathrm{Tr}^{\omega}_{k,2k-1}$.

Theorem 8. $\forall_{k\geq 1}\ \Theta_k(\mathcal{E}_k) = \mathcal{W}^{\omega}_{k,2k-1}$.

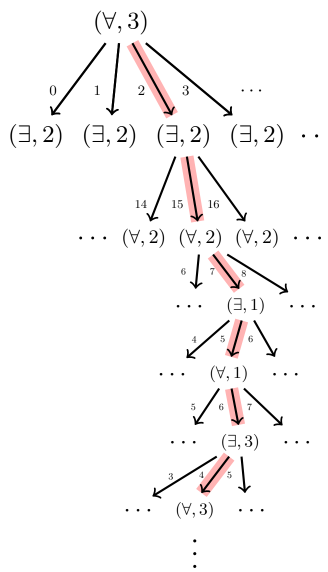

Fig. 7. An illustration of `treeMap`.

Proof. The proof is based on Matryoshka games. Consider a tree $t \in \mathrm{Tr}^{\omega}_{k,2k-1}$ and assume that a player $P \in \{\exists, \forall\}$ has a winning strategy σ on the tree t. We claim that P has a winning strategy in the Matryoshka game $\mathcal{G}_k$ with promise $\mathcal{E}_k$. From this fact the theorem will follow by an application of Corollary 2 and Proposition 6. For the simplicity we assume that $P = \exists$, the opposite case is analogous.

We will simulate the game on t in the Matryoshka game $\mathcal{G}_k$. A play in $\mathcal{G}_k$ consists of playing pairs of numbers (corresponding to moves in t) in the copies of $\mathcal{G}_0$ and additionally of deciding whether to *exit* an j-layer of the game or not. We say that a play in $\mathcal{G}_k$ is *fair* if whenever the players encounter a priority $k+j$ in t then they exit exactly j first layers of $\mathcal{G}_k$ (i.e. the layer $j+1$ is reached) and if they encounter a symbol $\bot$ or $\top$ then the players exit all the layers of $\mathcal{G}_k$.

Let $\exists$ use the original strategy σ in the copies of $\mathcal{G}_0$ and play "fairly" as long as $\forall$ does. If $\forall$ also plays "fairly" then the play is winning for $\exists$: either $\top$ is reached in t and $\exists$ wins since $t \in E_s$ or the play is infinite and $\exists$ wins by the parity condition — the priorities visited in $\mathcal{G}_k$ agree with those visited in t, see (7).

If $\forall$ does not play "fairly" (i.e. when a priority $k+j$ is reached in t he does not exit the l-layer of $\mathcal{G}_k$ with $l \leq j$ or he exits the $(j+1)$-layer of $\mathcal{G}_k$) then $\exists$ uses the following counter-strategy: whenever possible she exits the current layer of $\mathcal{G}_k$. There are two possible developments of such a play. The first case is that $\forall$ allows to exit the whole game and then $\exists$ wins thanks to (8). Now assume that $\forall$ never allows the game to reach a terminal position. In that case, let j be maximal such that the j-layer of $\mathcal{G}_k$ is visited infinitely often. By (7) we know that the limes superior of the priorities visited in $\mathcal{G}_k$ is $k+j-1$ and since $\forall$ is the owner of the vertices in the j-layer of $\mathcal{G}_k$ so $k+j-1 \equiv 0 \pmod 2$. Therefore, $\exists$ wins the play by the parity condition. □

As a consequence of Corollary 3 and Theorem 8 we obtain the desired completeness result.

Theorem 3. *$\mathcal{W}_{k-1,2k-1}$ is complete for the k-th level of the hierarchy of $\mathcal{R}$-sets.*

We note that this implies that every game tree language $\mathcal{W}_{i,k}$ is an $\mathcal{R}$-set belonging to the finite levels of the $\mathcal{R}$-hierarchy. Thus, by application of Kolmogorov's results (Theorem 6), we have obtained an alternative proof of Theorem 1 on the measurability of $\mathcal{W}_{i,k}$.

Remarks. The notion of $\mathcal{R}$-sets is a robust concept and admits natural variations. One can equivalently work in arbitrary (not zero-dimensional) Polish spaces and start from a basis of, e.g. Borel sets rather than clopens. The family of operations $\Theta_k = (\text{co-}\mathcal{R})^k(\bigcup \circ \bigcap)$ can be replaced by, e.g. either $(\text{co-}\mathcal{R})^k(\bigcup)$ or $(\text{co-}\mathcal{R})^k(\bigcap)$. Similarly, one can consider binary rather than countably branching, Matryoshka games. The notion of $\mathcal{R}$-sets remains unchanged in these alternative setups.

6 A remark on continuity of measures on $\mathcal{W}_{i,k}$

As we mentioned in the Subsection 1.3 of the Introduction, a natural method of proving continuity would be through an application of the *Boundedness Principle* (see, e.g. Section 34.B in [24]). In this Section we verify that indeed the method works for $\mathcal{W}_{0,1}$ but not for $\mathcal{W}_{1,3}$ or higher indices.

6.1 The Boundedness Principle for $\mathcal{W}_{0,1}$

In this section we prove the statement of Theorem 2 for $\mathcal{W}_{0,1}$, i.e. for the particular case of $i = 0$ and $k = 1$.

Consider an Borel measure μ such that $\mu(\mathcal{W}_{0,1}) > 0$. We will prove the following proposition.

Proposition 7. *For every Borel set $G \subseteq \mathcal{W}_{0,1}$, there exists a countable ordinal $\alpha < \omega_1$ such that $G \subseteq \mathcal{W}_{0,1}^{\alpha}$.*

The desired continuity property then follows from the above proposition as follows. For an arbitrary Borel measure μ, let $G \subseteq \mathcal{W}_{0,1}$ be a Borel set such that $\mu(G) = \mathcal{W}_{0,1}$. Such a set G exists since $\mathcal{W}_{0,1}$ is a measurable set. Then $\mu(G) \leq \mu(\mathcal{W}_{0,1}^{\alpha}) \leq \mu(\mathcal{W}_{0,1}) = \mu(G)$.

The rest of this section is devoted to the proof of Proposition 7.

For a tree $t \in \mathrm{Tr}_{0,1}$ we define

$$\mathsf{rank}(t) = \min\{\alpha < \omega_1 \mid t \in \mathcal{W}_{0,1}^{\alpha}\}$$

or $\mathsf{rank}(t) = \omega_1$ if the minimum is not well–defined. Note that since by Theorem 5 the equality $\mathcal{W}_{i,k} = \bigcup_{\alpha<\omega_1} \mathcal{W}_{0,1}^{\alpha}$ holds, $\mathsf{rank}(t) = \omega_1$ if and only if $t \notin \mathcal{W}_{0,1}$.

We now establish the following technical fact.

Proposition 8. *For $i = 0$ and $k = 1$ the rank* rank *is a co-analytic rank ([24, §34.B]), that is there exist an co-analytic relation $\leq^{\mathbf{\Pi}^1_1} \subseteq \mathrm{Tr}_{0,1} \times \mathrm{Tr}_{0,1}$ and an analytic relation $\leq^{\mathbf{\Sigma}^1_1} \subseteq \mathrm{Tr}_{0,1} \times \mathrm{Tr}_{0,1}$ such that for every $s \in \mathrm{Tr}_{0,1}$ and $t \in \mathcal{W}_{0,1}$ holds*

$$\mathsf{rank}(s) \leq \mathsf{rank}(t) \;\Leftrightarrow\; s \leq^{\mathbf{\Pi}^1_1} t \;\Leftrightarrow\; s \leq^{\mathbf{\Sigma}^1_1} t.$$

Proof. This technically looking statement actually follows quite easily either through a direct definition of appropriate relations $\leq^{\mathbf{\Pi}^1_1} \subseteq \mathrm{Tr}_{0,1} \times \mathrm{Tr}_{0,1}$, $\leq^{\mathbf{\Sigma}^1_1} \subseteq \mathrm{Tr}_{0,1} \times \mathrm{Tr}_{0,1}$ on $\mathrm{Tr}_{0,1}$ or through application of the Borel derivative method, see [24, Section 34.D and Theorem 34.10]. We decide below for the second method, because it is conceptually simpler. Formally, a derivative works on partial trees (that is, restrictions of $t \in \mathrm{Tr}_{0,1}$ to certain subsets of the binary tree) and assigns to a partial tree t another partial tree $D(t) \subseteq t$.

First note that the set D_0 of partial trees t on which $\exists$ has a winning strategy that visits at most once priority 1 and either ends up in a finite vertex without extensions in t or wins in the infinity in the usual sense of the parity condition, is a Borel subset of partial trees: by König's lemma it is enough to have longer and longer finite (i.e. specified up–to a finite level of the tree) strategies visiting 1 at most once. Thus

$$D_0 = \{t \mid \bigcap_n \{t \downarrow_n \in F_n\}\}$$

where $t \downarrow_n$ denote the partial tree obtained by removing all the vertices of t of depth $\geq n$ and F_n is the set of partial trees of depth at most n which have a strategy $\sigma_\exists$ for $\exists$ such that any play consistent with $\sigma_\exists$ visits 1 at most one. The sets F_n are clearly Borel sets, hence D_0 is Borel.

Now, we define the derivative D as follows: D takes as input a partial tree t and returns the partial tree obtained by removing from t the vertex $w \in \mathsf{dom}(t)$, and all the descendants of w in t, for all w such that t_w (the subtree of t rooted at w) is in D_0.

Clearly, such a derivative is decreasing (i.e. $D(t) \subseteq t$, for all t). Thus, by iterating D on input t, we eventually reach a fixed-point t':

$$t \subseteq D(t) \subseteq D(D(t)) \subseteq \ldots t' = D(t')$$

We then observe that $t' = \epsilon$ if and only if $t \in \mathcal{W}_{0,1}$. The number of interactions of D to t until reaching ϵ (denoted $|t|_D$ in [24, Section 34.D]) is exactly $\mathsf{rank}(t)$. Also, as a mapping from partial trees to partial trees, D is clearly a Borel function (because D_0 is Borel). Hence D is a Borel derivative and from [24, Theorem 34.10] follows that rank is a $\mathbf{\Pi}^1_1$-rank. □

Since rank is a co-analytic rank on $\mathcal{W}_{0,1}$, the statement of Proposition 7 is an instance of the Boundedness Principle ([24, Theorem 35.23]).

6.2 Failure of the boundedness principle for higher ranks

In this section we show that the method from Section 6.1 does not generalize to higher indices. Namely, we will prove the following fact.

Proposition 9. *There exists a Borel (actually closed) set* $G \subseteq \mathcal{W}_{1,3}$ *such that for all countable ordinals* $\alpha < \omega_1$ *it holds that* $G \nsubseteq \mathcal{W}^{\alpha}_{1,3}$.

The rest of this section contains the proof of Proposition 9.

Proof. For a tree $t \in \mathrm{Tr}_{1,2}$ let us denote with tree $\bar{t} \in \mathrm{Tr}_{2,3}$ the *dual* tree, obtained by replacing a label (P, i) by $(\bar{P}, i+1)$. Clearly, $t \in \mathcal{W}_{1,2}$ if and only if $\bar{t} \notin \mathcal{W}_{2,3}$.

Now, consider the set of trees $G \subseteq \mathrm{Tr}_{1,3}$ defined as $G = \{f(t) \mid t \in \mathrm{Tr}_{1,2}\}$, where $f\colon \mathrm{Tr}_{1,2} \to \mathrm{Tr}_{1,3}$ is defined as $f(t) \stackrel{\text{def}}{=} (\exists, 1)(t, \bar{t})$, as shown at Figure 6.2.

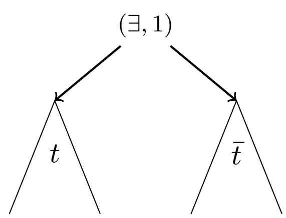

Fig. 8. The tree $f(t)$.

Note that $G \subseteq \mathrm{Tr}_{1,3}$ is a closed set since it is specified by a closed constraint: $t \in G$ if and only if for each vertex of the form dv for $d \in \{0,1\}$ and $v \in \{0,1\}^*$, if $t(dv) = (P, i)$ then $\bar{t}(dv) = (\bar{P}, i+1)$. In particular, G is a Borel set.

Observe that for each $t \in \mathrm{Tr}_{1,2}$ either $t \in \mathcal{W}_{1,2}$ or $\bar{t} \in \mathcal{W}_{2,3}$. Hence $f(t)$ is always a tree wining for $\exists$. Therefore $G \subseteq \mathcal{W}_{1,3}$.

Note that, if $t \in \mathcal{W}_{1,2}$ then $\exists$ can win in $f(t)$ by moving to the left subtree (i.e. t) and then play a winning strategy on t. Such a strategy does not visit any vertex having priority 3 and therefore, in accordance with the definition of $\mathcal{W}_{1,3}$, we have that $f(t) \in \mathcal{W}^{1}_{1,3}$.

Now consider, for an arbitrary countable ordinal α, a tree $\bar{t} \in \mathcal{W}_{2,3}$ such that $t \in \mathcal{W}^{\alpha}_{2,3}$ and such that $t \notin \mathcal{W}^{\beta}_{2,3}$ for every $\beta < \alpha$. Note that, $t \in \mathrm{Tr}_{2,3}$ is also a tree in $\mathcal{W}^{\alpha}_{1,3}$ (just not having occurrences of vertices with priority 1) and, by definition of $\mathcal{W}_{1,3}$, it also holds that $\bar{t} \in \mathcal{W}^{\alpha}_{1,3}$ and $\bar{t} \notin \mathcal{W}^{\beta}_{1,3}$, for all $\beta < \alpha$.

Since $\bar{t} \in \mathcal{W}_{2,3}$, we have that $t \notin \mathcal{W}_{1,2}$ and therefore any winning strategy for $\exists$ in $f(t)$ has to move to the right subtree (i.e. $\bar{t}$). Therefore it holds that $f(t) \in \mathcal{W}^{\alpha}_{2,3}$ and $t \notin \mathcal{W}^{\beta}_{2,3}$ for every $\beta < \alpha$.

Since α is an arbitrary countable ordinal, the proof of Proposition 9 is completed. □

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