Elementary Modal Logics – Decidability and Complexity

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I. Introduction

Modal logic was first introduced by philosophers to study the deductive behaviour of the expressions *it is necessary that* and *it is possible that*. Nowadays, it is widely used in several areas of computer science, including formal verification and artificial intelligence.

Syntactically, modal logic extends propositional logic by two unary operators: $\Diamond$ and $\Box$. The formal semantics is usually given in terms of Kripke structures. Basically, a Kripke structure is a directed graph, called a *frame*, together with a valuation of propositional variables. Vertices of this graph are called *worlds*. For each world truth values of all propositional variables are defined independently. In this semantics, $\Diamond\varphi$ means *the current world is connected to some world in which* φ *is true*; and $\Box\varphi$, equivalent to $\neg\Diamond\neg\varphi$, means φ *is true in all worlds to which the current world is connected*.

Basic modal logic, as defined above, is rather weak and has limited applications. Thus, numerous variations and extensions of this formalism, including temporal and description logics, have been proposed and investigated. They vary in the complexity of the satisfiability problem, which usually lies between NP and 2ExpTime. However, even some simple extensions may lead to undecidability (see e.g. [1]).

A popular approach to extend modal logic is to restrict the class of admissible Kripke frames. For example, we may require them to be reflexive and transitive (which corresponds to modal logic S4) or enforce their transition relation to be an equivalence relation (S5). There are several ways in which such restrictions can be imposed.

We focus on restricting the class of the admissible frames by a first-order logic sentence that uses a single binary relation R, which is interpreted as the *transition relation*. Eg., the sentence $\forall xyz.xRy \wedge yRz \Rightarrow xRz$ defines the class of all transitive frames. Modal logic over a class of frames definable by a first-order logic sentence is called *an elementary modal logic*.

The main goal of our research is to classify all elementary modal logics with respect to decidability and complexity of their satisfiability problems.

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II. Plan of the talk

The aim of the Highlights conference talk is to present a wide picture of the latest research on elementary modal logics definable by universal first-order formulae. The talk will provide all the necessarily definitions, discuss the motivation, summarise the recent research papers on this topic, and discuss the open problems. Below we present a short summary of the recent results.

Elementary modal logics defined by universal first-order sentences may be undecidable [2], even if we admit only sentences with three variables [3]. This holds regardless of whether we consider *local* satisfiability (is a given modal formula φ true at *some* world of some structure?) or *global* satisfiability (is φ true at *all* worlds of some structure?), as well as if we consider *finite* or *general* satisfiability (i.e. if we restrict admissible structures to be finite or not). In [4], however, it was proved that there are elementary modal logics with decidable general satisfiability problem and undecidable finite satisfiability problem and the same the other way round.

On the positive side, for the elementary modal logics definable by universal Horn formulae, the satisfiability problem is always decidable, both in the general case [5] and in the finite case [6]. The most recent result [7] may be considered as a first step towards *elementary temporal logics* – it was shown that universally definable elementary modal logics are always decidable under the assumption that the transition relation is transitive. The positive results comes usually with the precise complexity bounds.

References

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