The Ultimate Undecidability Result for the Halpern–Shoham Logic

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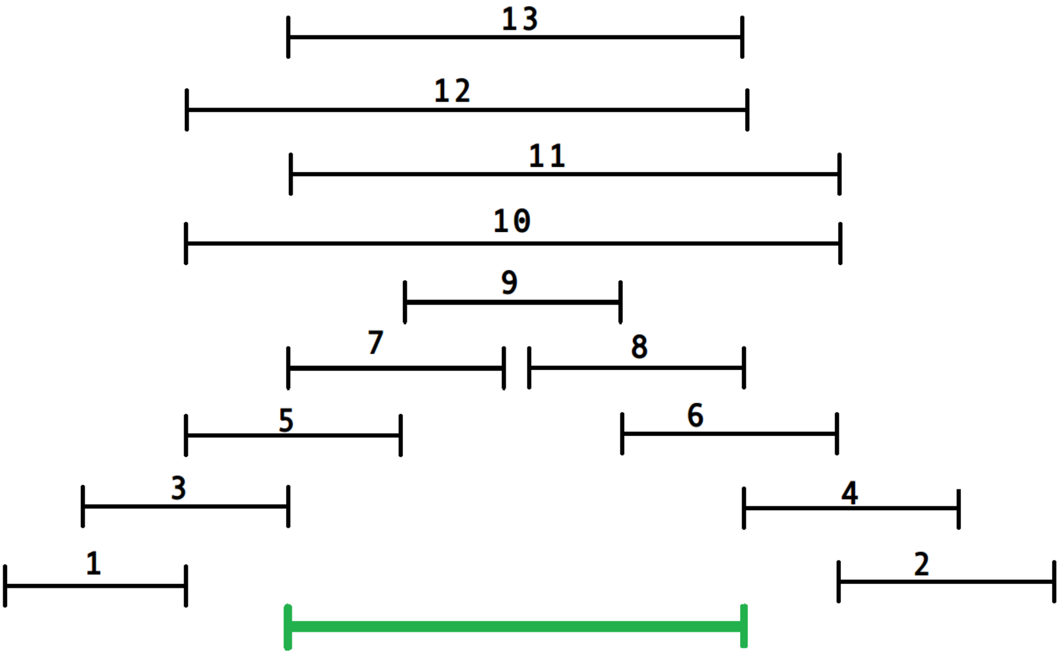
Intervals

- Given an arbitrary total order $\langle \mathbb{D}, \leq \rangle$.
- An interval: $[a, b]$ such that $a, b \in \mathbb{D}$ and $a \leq b$.
- What relative positions of two intervals can be expressed using $\leq$?

13 relative positions of intervals can be expressed using $\leq$



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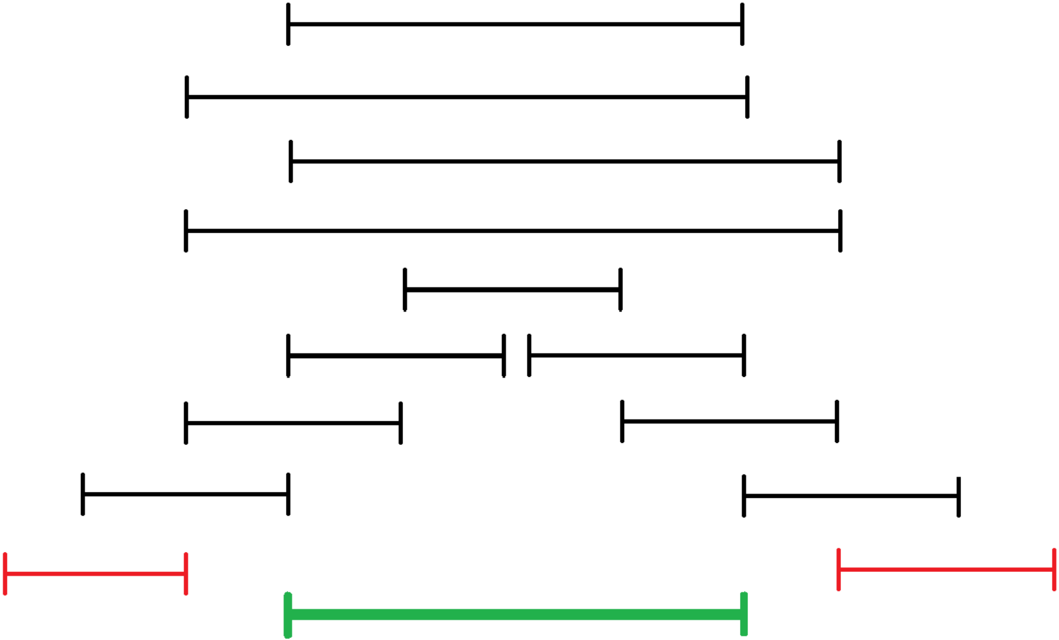
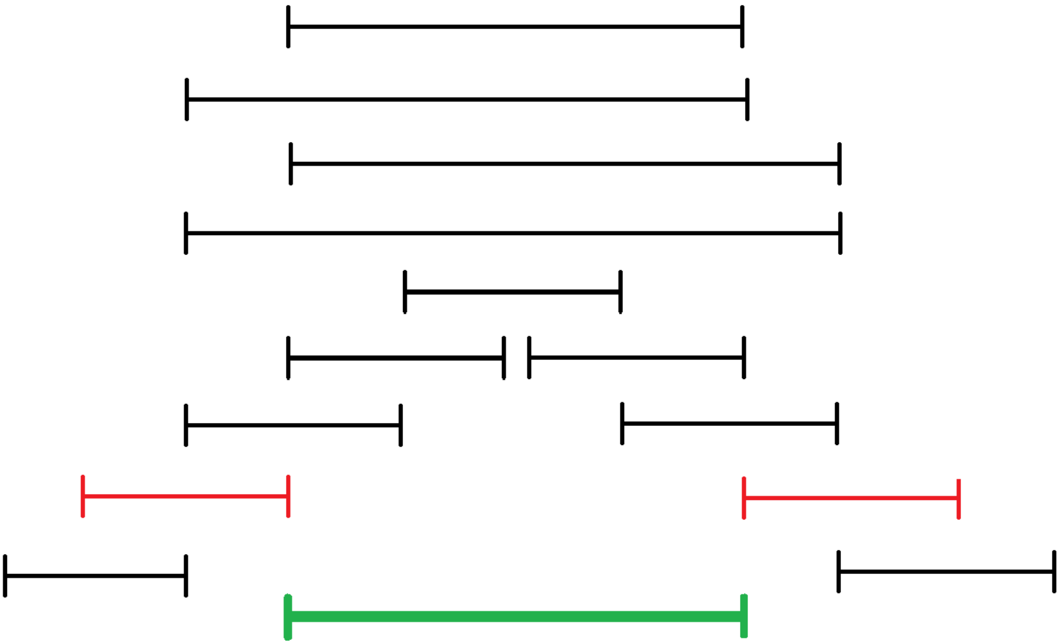


Before and after.

13 relative positions of intervals can be expressed using $\leq$



Meet and met by.

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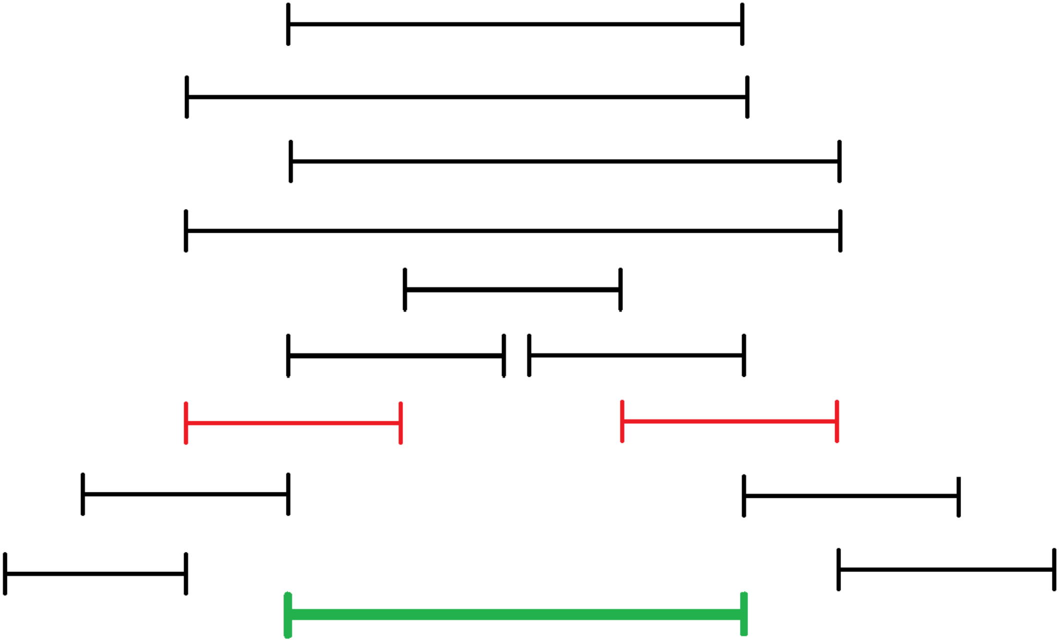

Overlaps and overlapped by.

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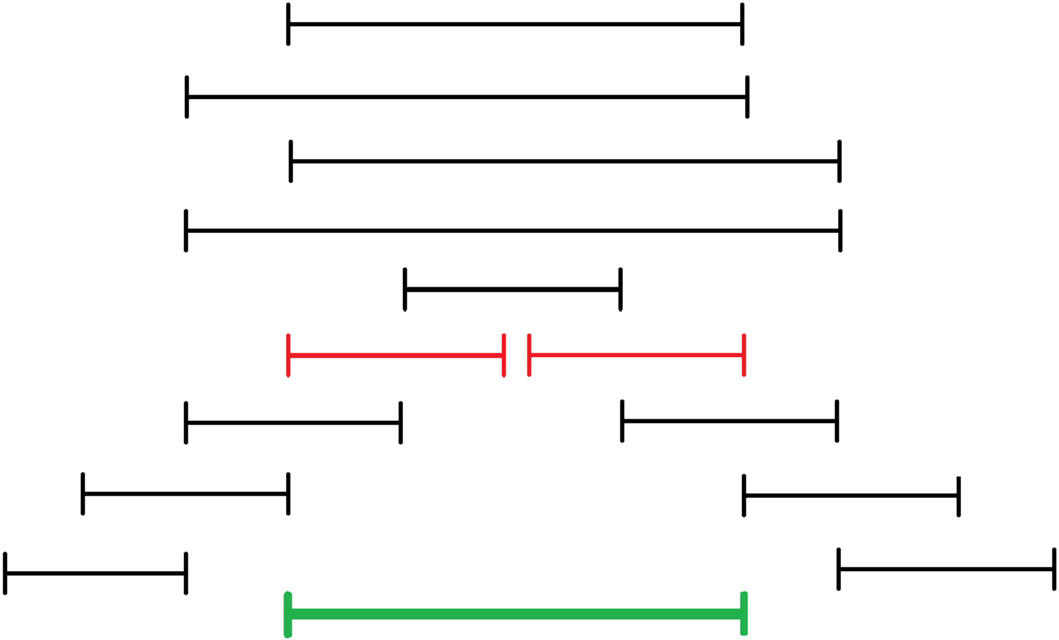

Starts and finishes.

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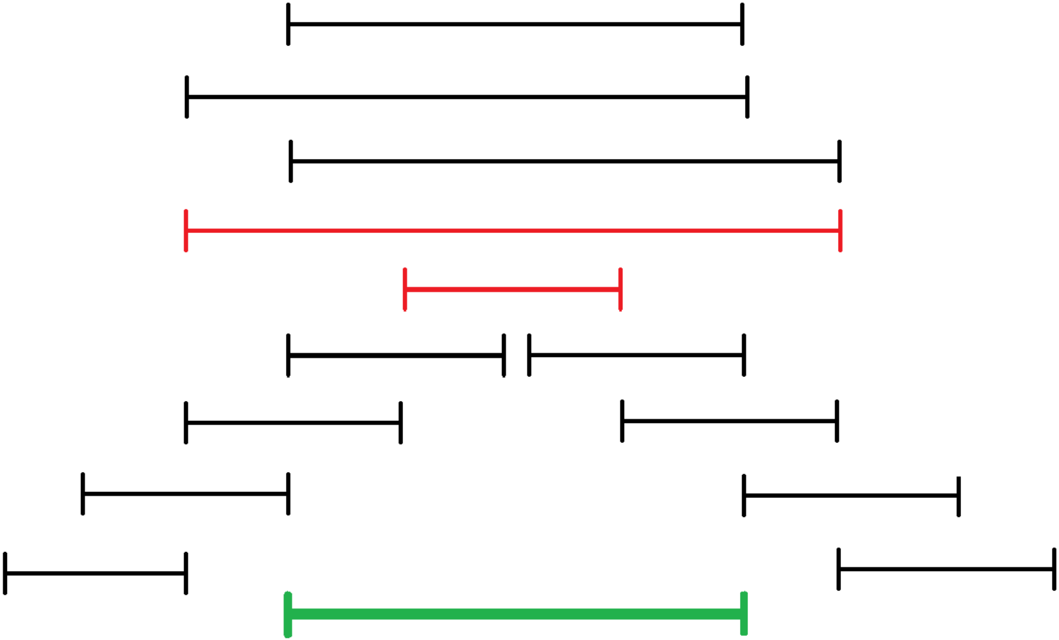

Contains and during.

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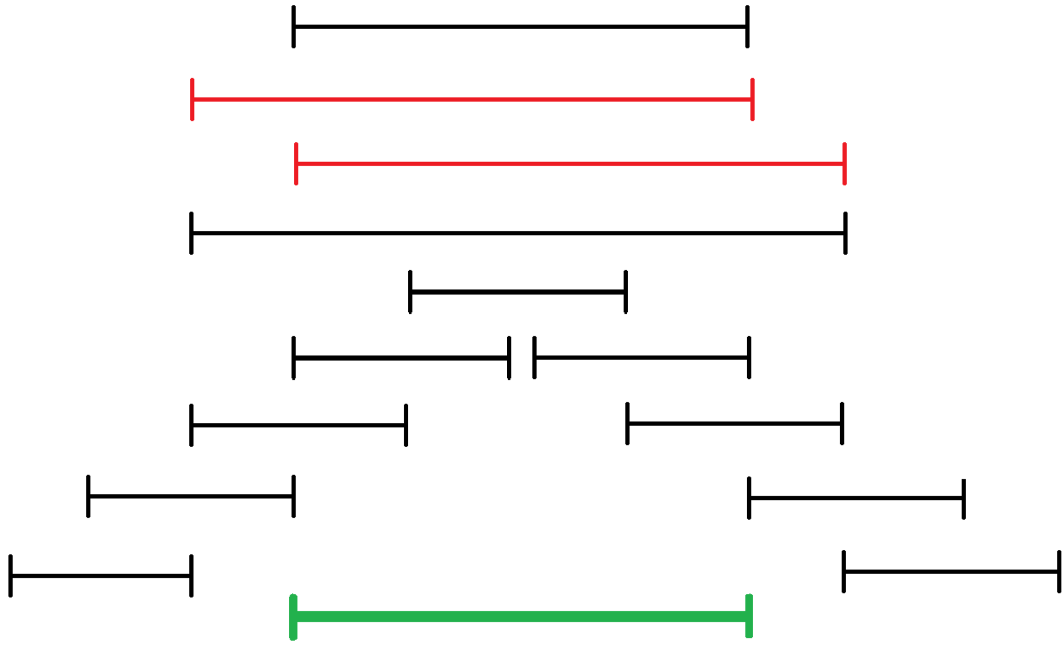

Started by and finished by.

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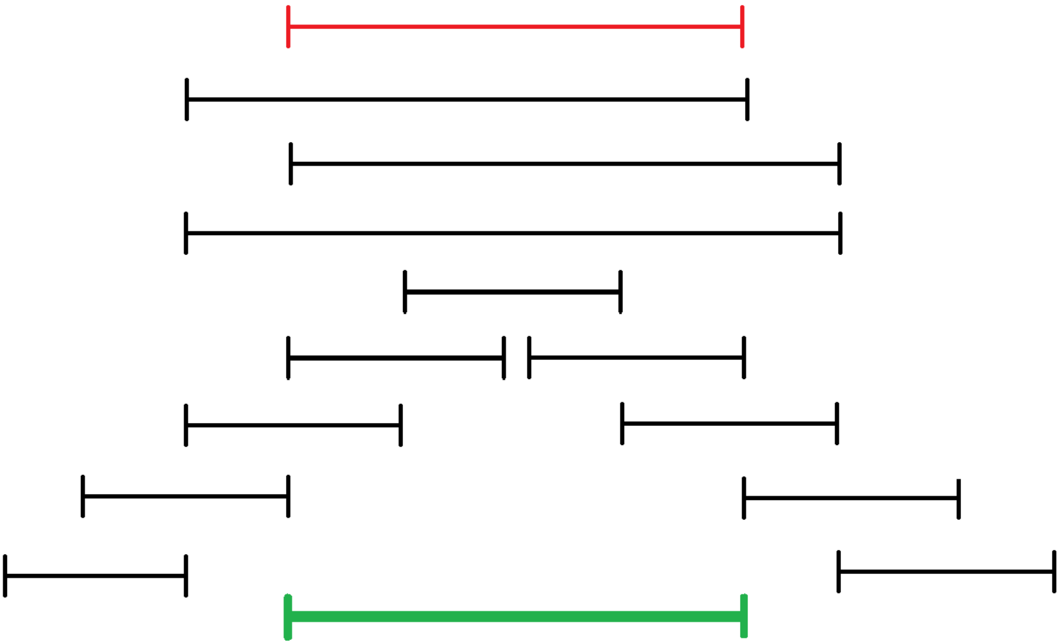
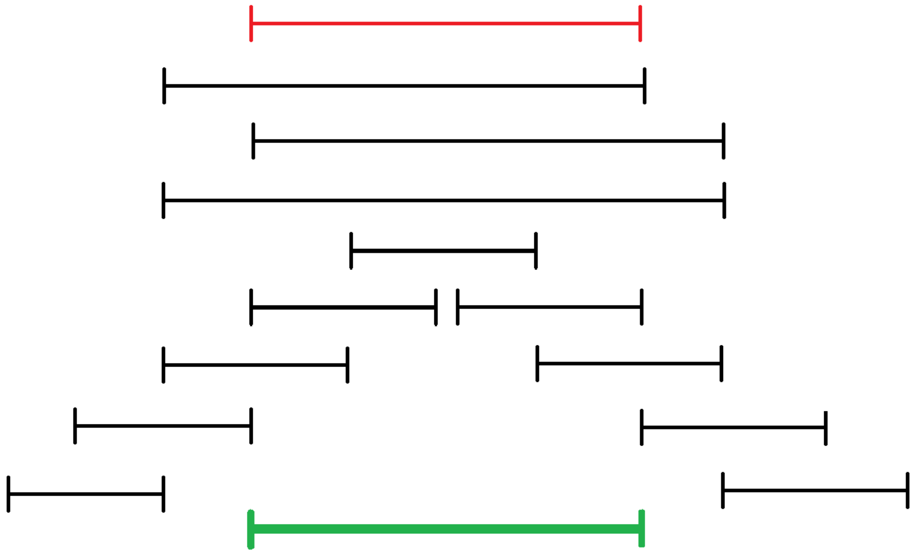


Equals.

What can we do with those relations?



Allen's algebra.

$$\forall xy.x \textit{ before } y \Rightarrow \exists z.z \textit{ meet } y \wedge z \textit{ met by } x$$

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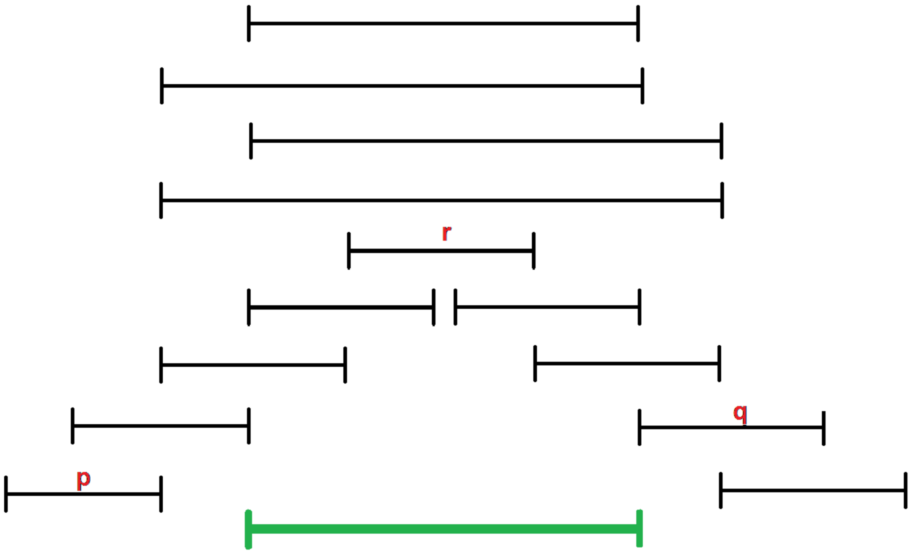

Halpern–Shoham logic.

$$\langle before \rangle p \wedge \langle during \rangle (r \wedge \langle after \rangle q)$$

Formally - the models

- Any total order $\mathcal{D} = \langle \mathbb{D}, \leq \rangle$
- Set of propositional variables $\mathcal{V}ar$
- Classic temporal logics — labeling $\gamma : \mathbb{D} \rightarrow \mathcal{P}(\mathcal{V}ar)$

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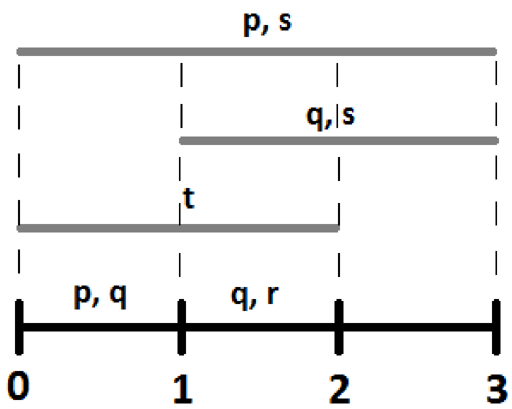
- Interval temporal logics — labeling $\gamma : \mathrm{I}(\mathbb{D}) \to \mathcal{P}(\mathcal{V}ar)$,

  where $\mathrm{I}(\mathbb{D}) = \{[a,b] | a, b \in \mathbb{D} \wedge a \leq b\}$



The Halpern–Shoham Logic

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The Logic of Subintervals

- The logic of subintervals contains only one operator (D):
- $\langle D\rangle\varphi$ is satisfied if φ is satisfied in some subinterval.
- $[D]\varphi$ is satisfied if φ is satisfied in all subintervals.

Logic of subintervals results

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What can we express using the “subinterval” relation?

- “each morning I spend a while thinking of you”

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- “each nice period of my life contains an unpleasant fragment”
- “there is no error while printing”

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- The logic of subintervals is **undecidable** over the class of discrete orders (this presentation).

Our undecidability result

Assumptions

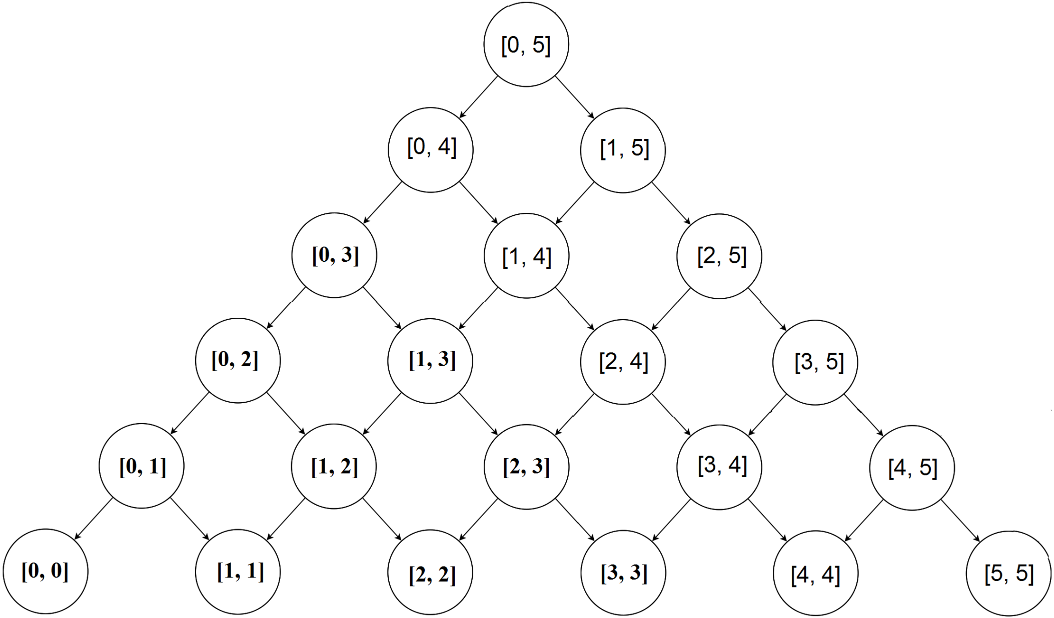
Assumption	In our paper	In this presentation
Order	All discrete	All finite
Do we allow point intervals ($[a, a]$)?	Whatever	Yes
Subinterval relation or superinterval relation?	Does not matter	Subinterval

An overview of the proof

The logic of subintervals is **undecidable**

over the class of all finite structures.

How we imagine that — "triangle structures"



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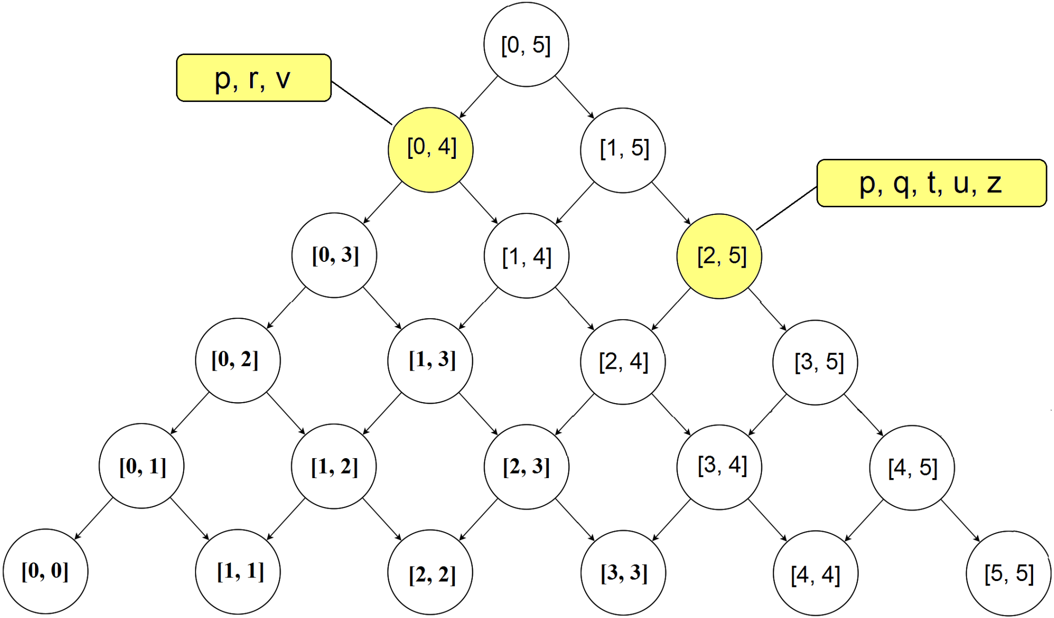

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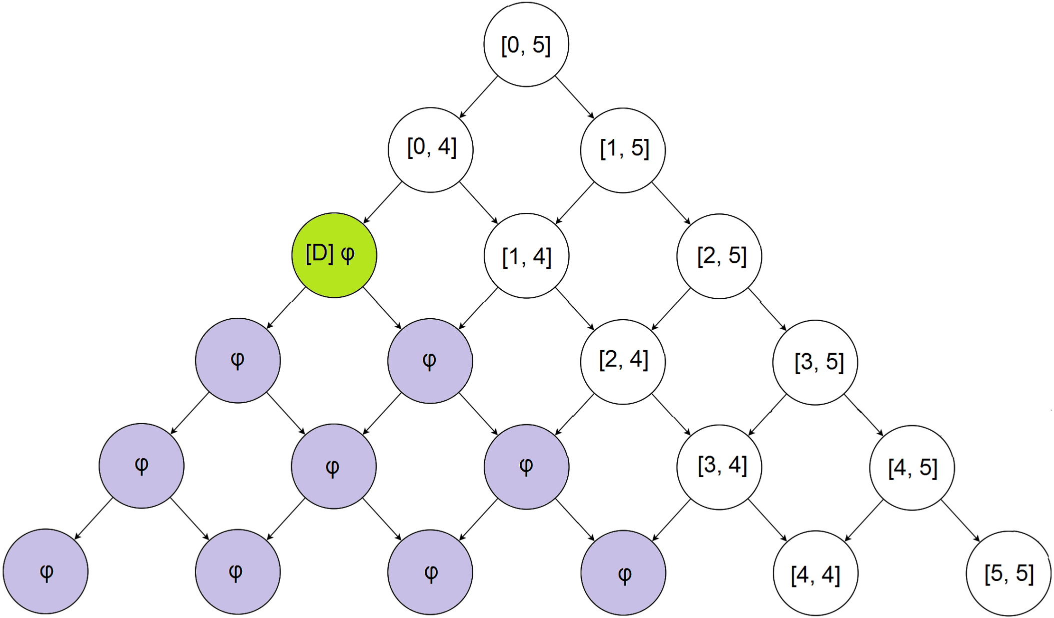
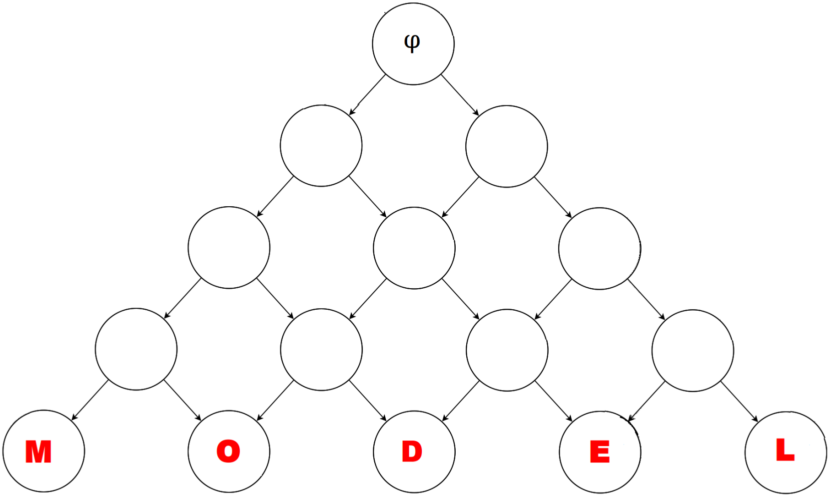


Problem 1 — symmetry



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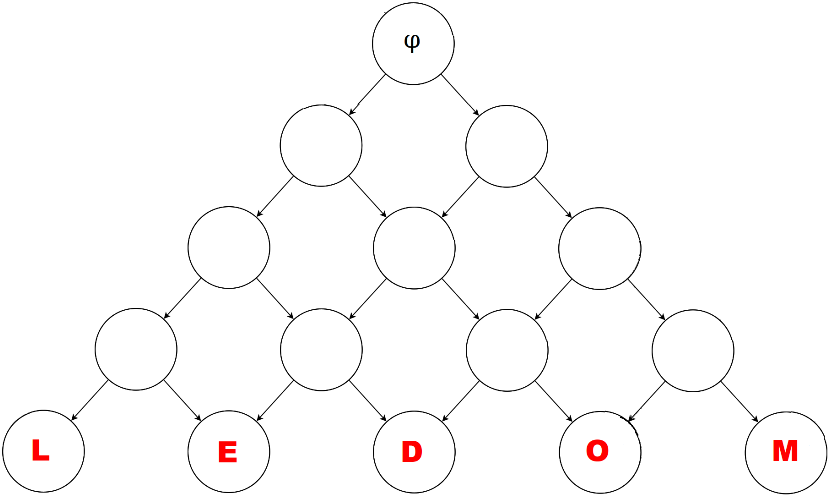
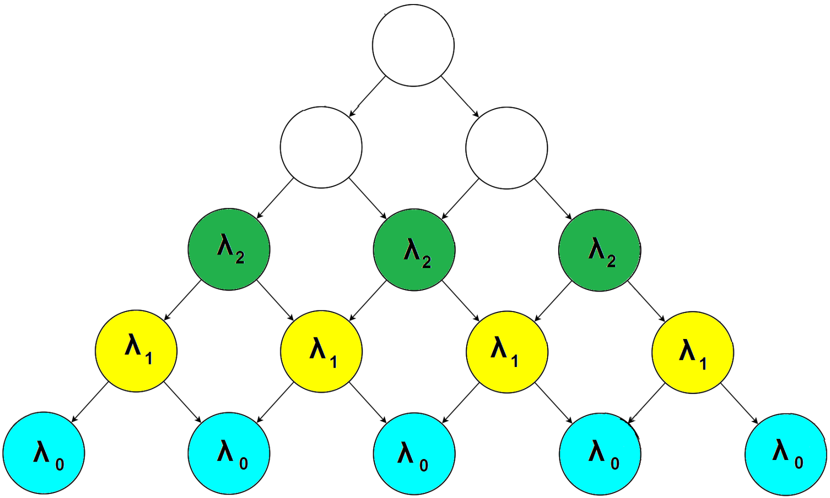


Our little library of formulae

- λ_0: $[D]\bot$



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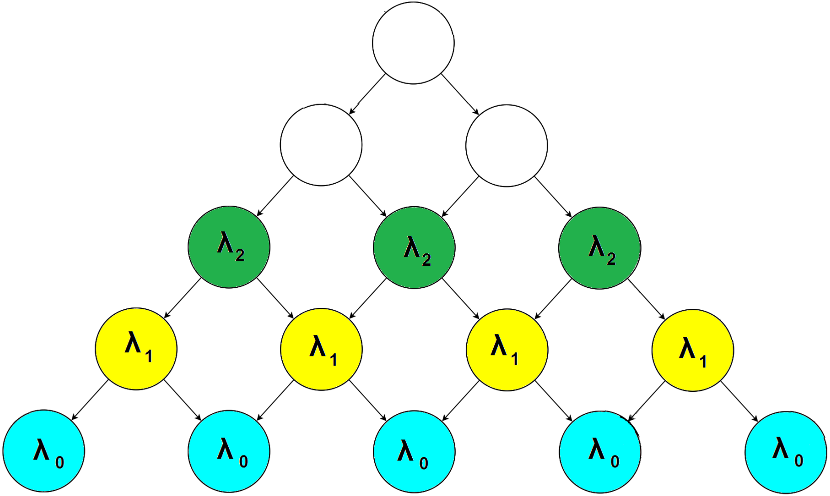

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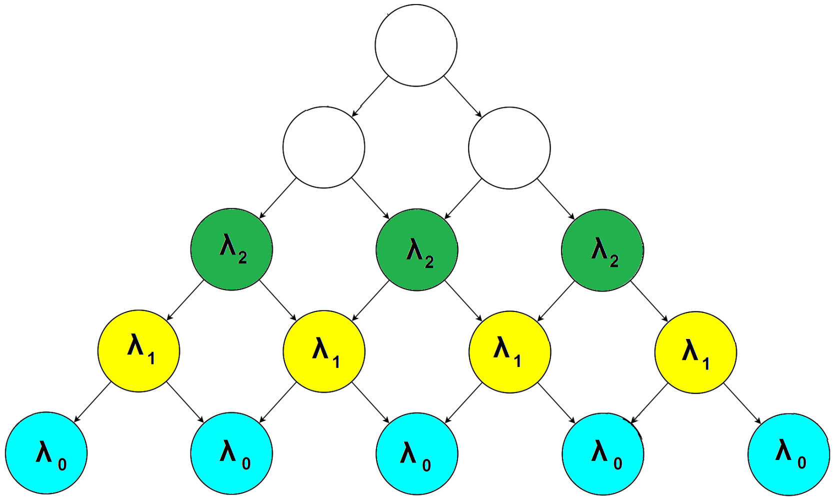

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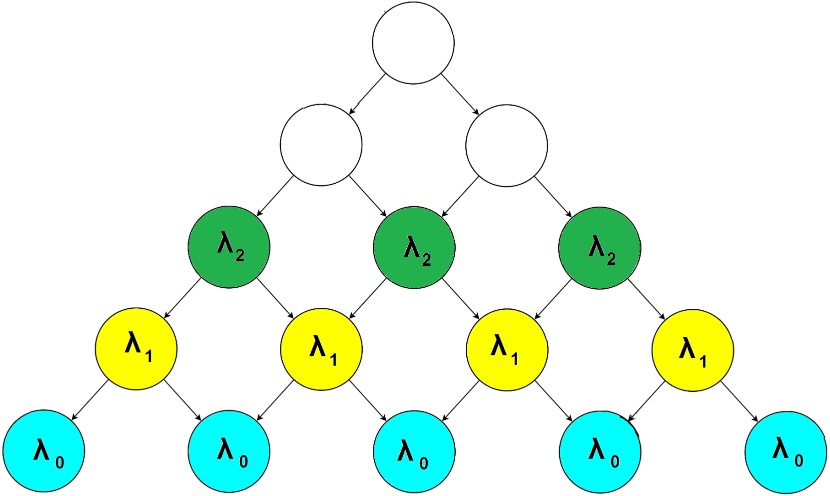
- λ_1: $\lambda_{\leq 1} \wedge \neg\lambda_0$



Problem 1 — symmetry

Now we can deal with the symmetry

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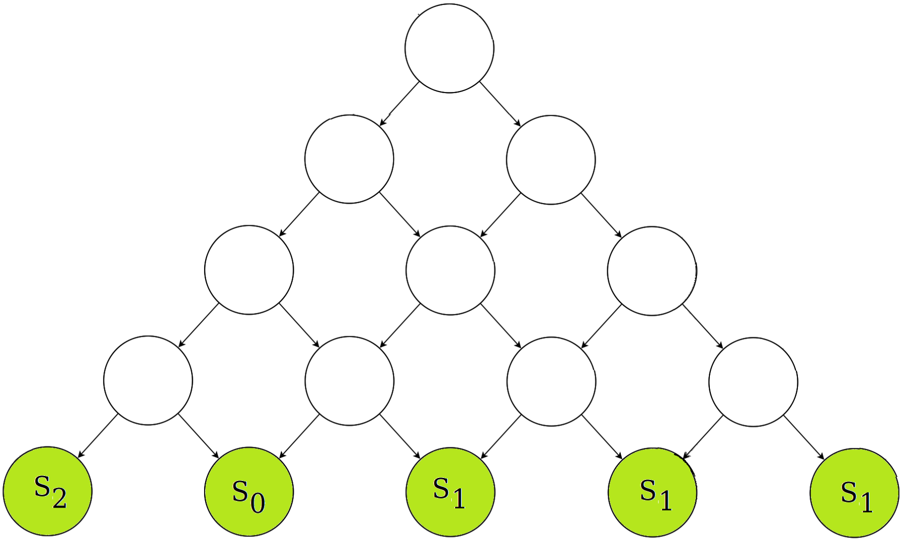

Each point-interval is labeled with exactly one of s_0, s_1, s_2, and each interval labeled with s_0, s_1, or s_2 is an point-interval.

$$[D](\lambda_0 \Leftrightarrow s_1 \vee s_2 \vee s_0) \wedge [D]\neg(s_0 \wedge s_1) \wedge [D]\neg(s_1 \wedge s_2) \wedge [D]\neg(s_0 \wedge s_2)$$

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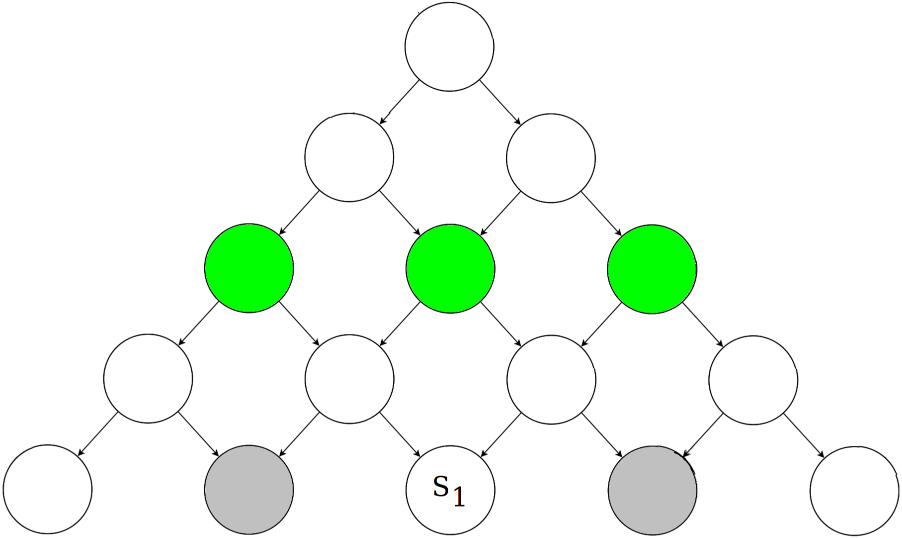

Each interval with length 2 contains intervals labeled with s_0, s_1, and s_2.

$$[D](\lambda_2 \Rightarrow \langle D \rangle s_0 \wedge \langle D \rangle s_1 \wedge \langle D \rangle s_2)$$

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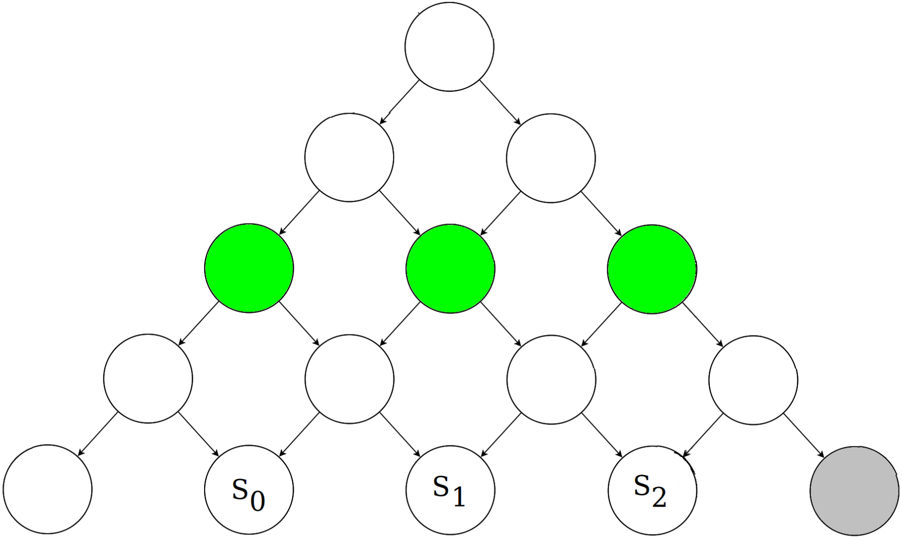

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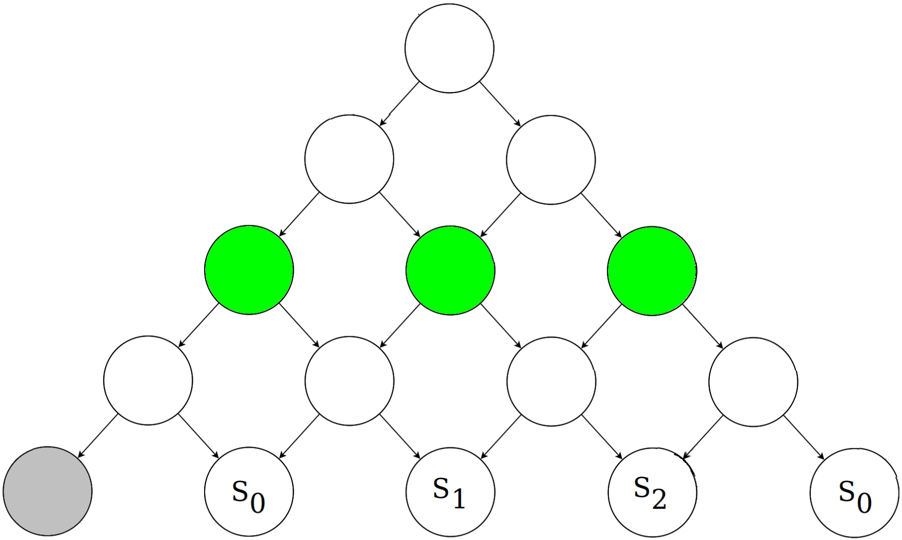

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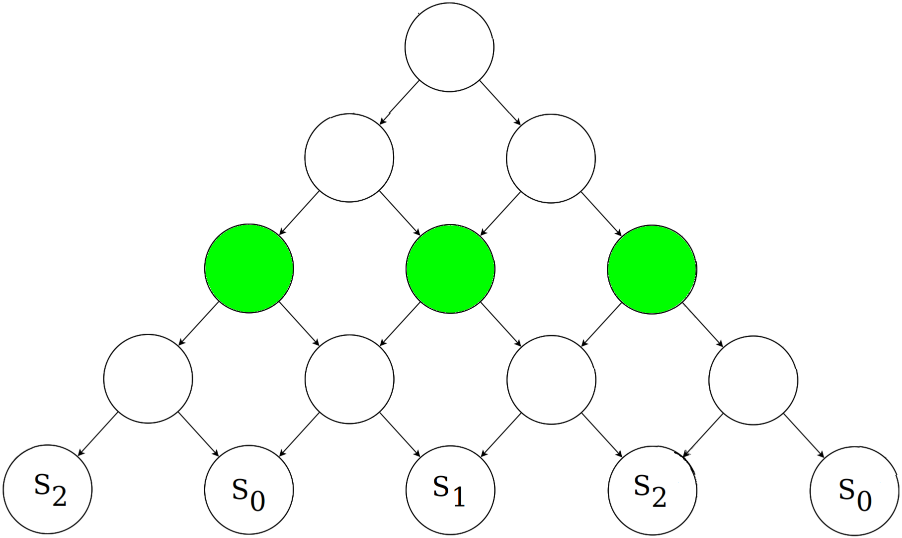

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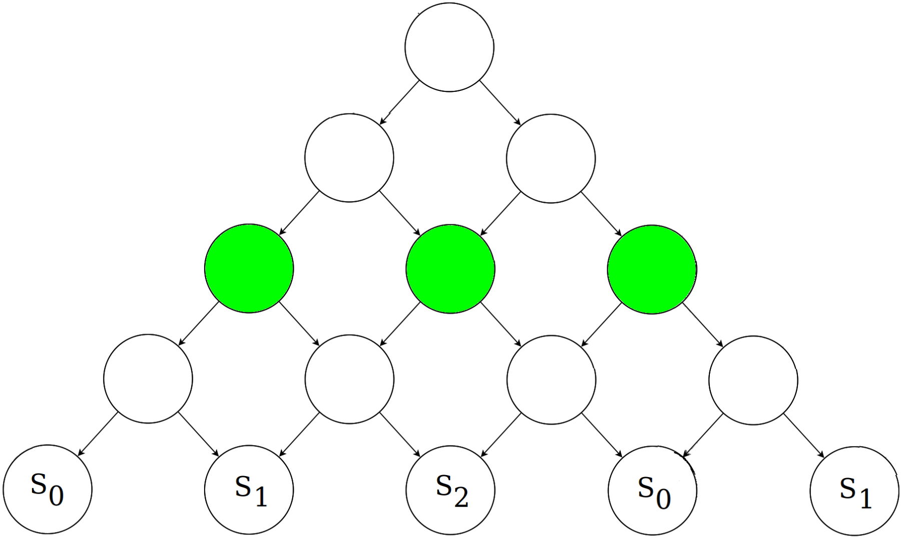

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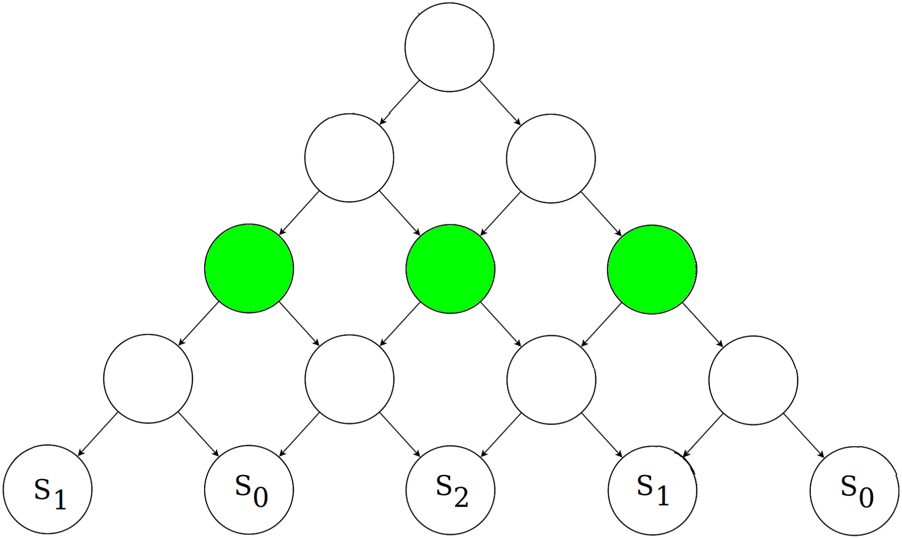

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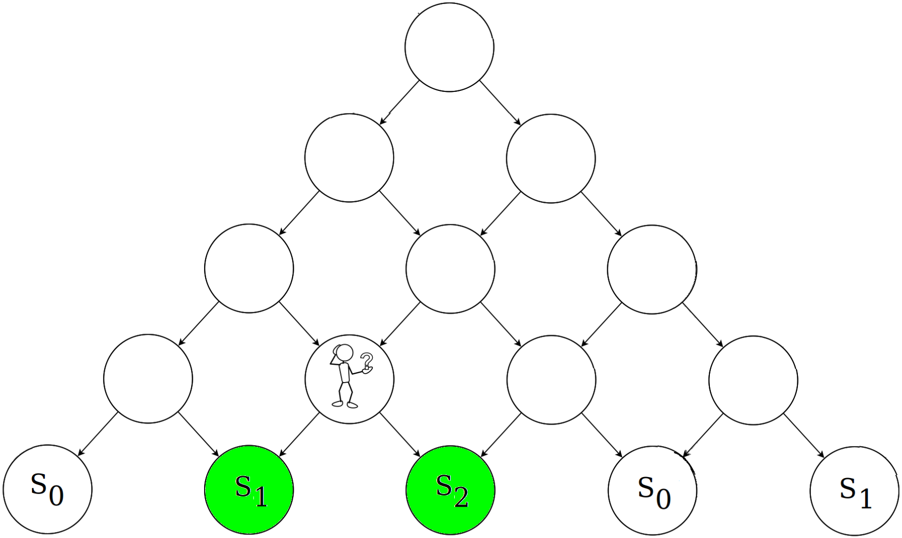


The source of the undecidability

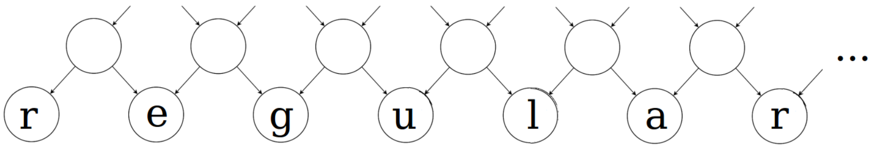
Regularity + ability to measure $\geq$ undecidability

The source of the undecidability

$$\text{Regularity} + \text{ability to measure} \geq \text{undecidability}$$

Actually, with D we only have very limited ability to measure. One of the technical lemmas is that this limited ability already leads to the undecidability.

Second step – regularity



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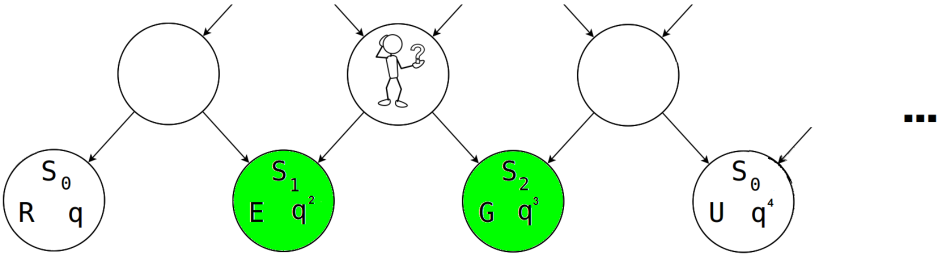

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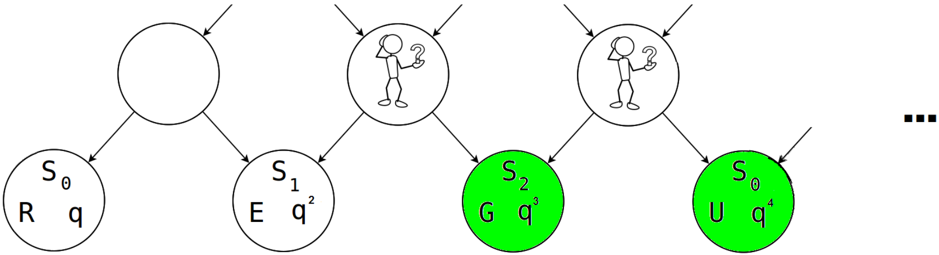

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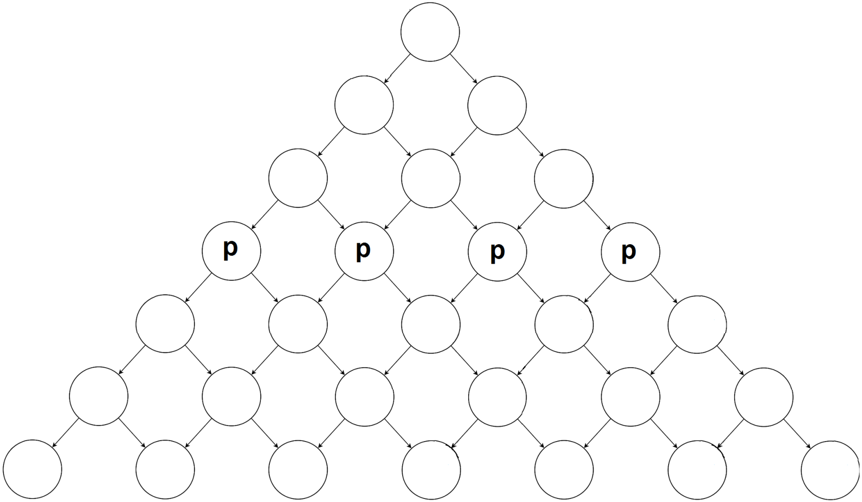
Third step — the cloud.



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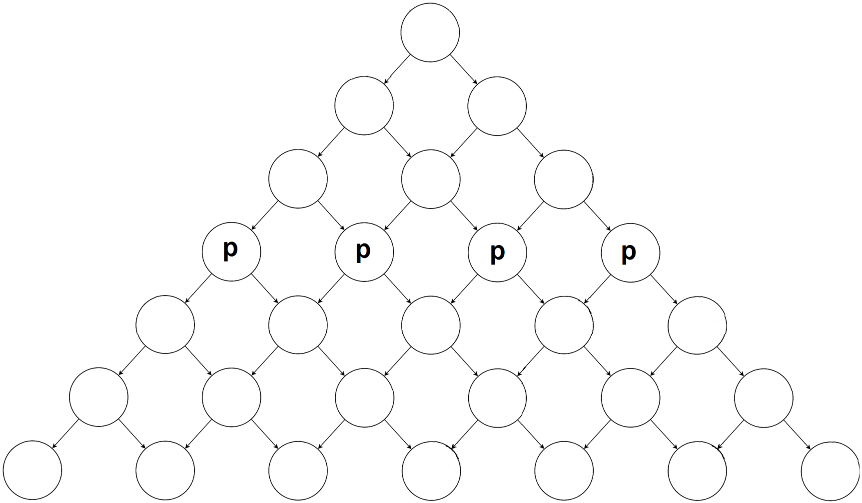
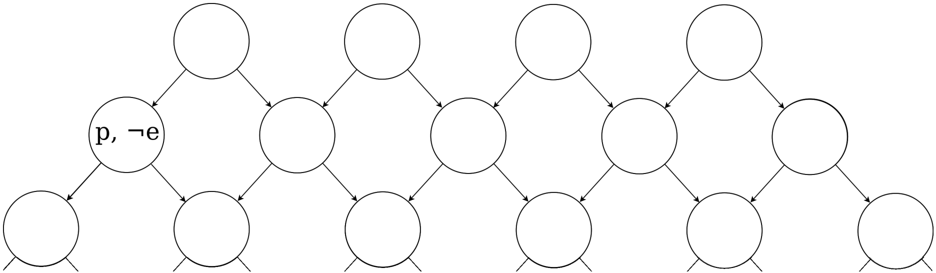


What remains to be explained:

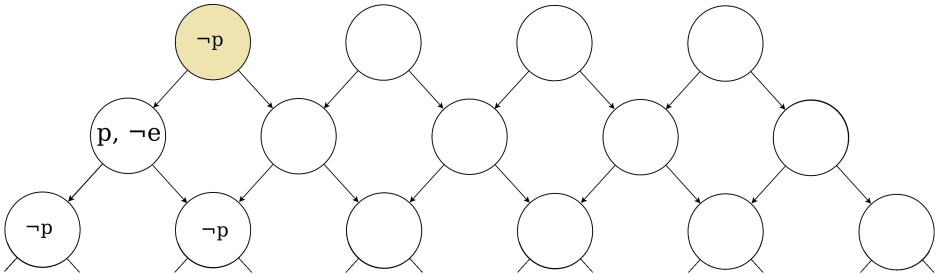
1. How to write a formula saying that a propositional variable p is a cloud.
2. How to use this cloud.

Third step — the cloud.



- There exists an interval labeled with p.
- Intervals labeled with p do not contain each other.
- Any interval that contains an interval with p, contains two such intervals (one with e and one with $\neg e$).

Third step — the cloud.



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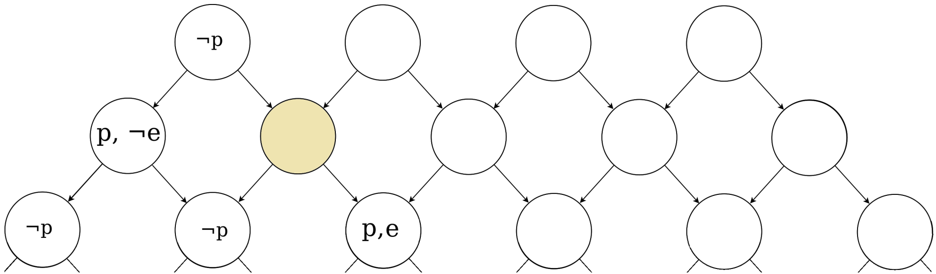

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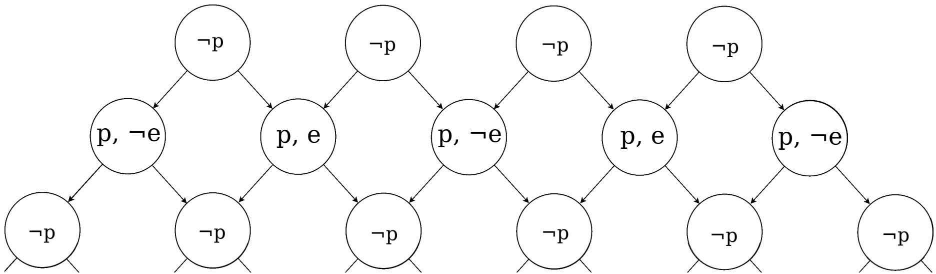

- There exists an interval labeled with p.
- Intervals labeled with p do not contain each other.
- Any interval that contains an interval with p, contains two such intervals (one with e and one with $\neg e$).

Third step — the cloud.

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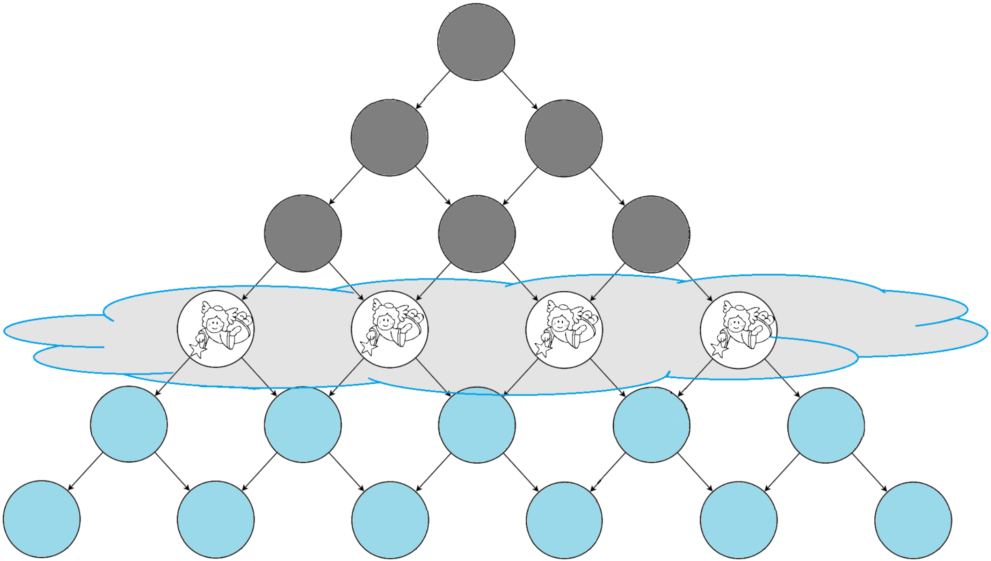
1. $\langle D \rangle p$
2. $[D](p \Rightarrow [D]\neg p)$
3. $[D](\langle D \rangle p \Rightarrow \langle D \rangle (p \wedge e) \wedge \langle D \rangle (p \wedge \neg e))$

Toy example

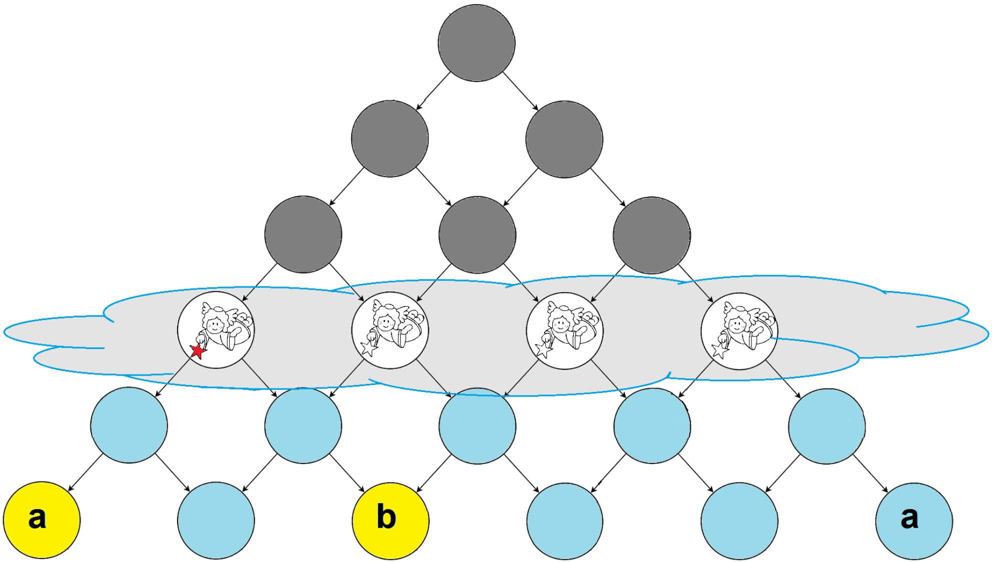
- We already know how to encode regularity.
- Consider the regular language defined by $(ac^*bc^*)^*$.
- We want to force that each maximal block of c has the same length.

Toy example



- No angel can see both *a* and *b*.
- Each angel has to see *a* or *b*.

Toy example



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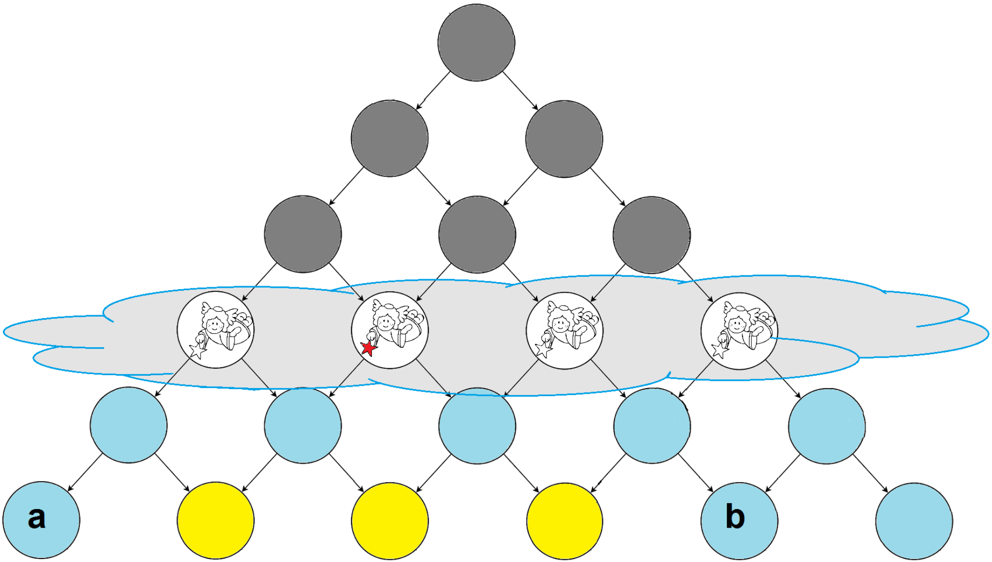

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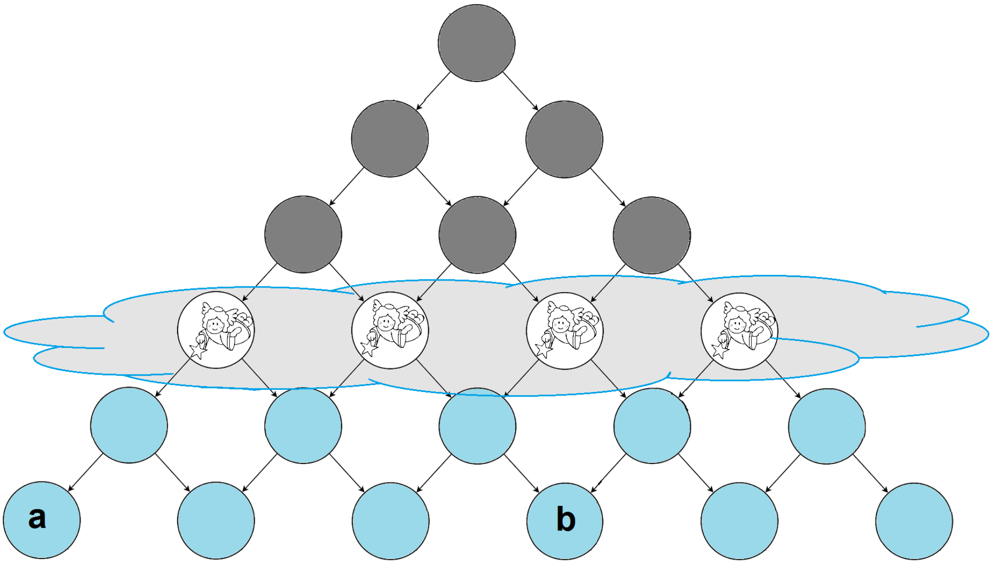

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Summary

- The logic of subintervals is **decidable** over the class of **dense** structures.
- The logic of subintervals is **undecidable** over the class of **discrete** structures.

Summary

- The logic of subintervals is **decidable** over the class of **dense** structures.
- The logic of subintervals is **undecidable** over the class of **discrete** structures.
- Is the logic of subintervals decidable over the class of all structures? (open)

Thank you!

Thank you!

Coming next: Davide Bresolin, Angelo Montanari, Pietro Sala and Guido Sciavicco. What's decidable about Halpern and Shoham's interval logic? The maximal fragment $AB\bar{B}L$