2-SAT Problems in Some Multi-valued Logics Based on Lattices

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***Abstract*— We prove that regular 2-SAT with signs of the form $\uparrow i$ and $\downarrow i$, where underlying truth value set forms a lattice, is solvable in quadratic time in the size of the input, and in the case where the lattice is fixed, in linear time in the size of the formula. Moreover, we show that the satisfiability problem for 2-CNF formulas in multi-valued logics based on arbitrary De Morgan algebras may be done in time linear in the size of the formula and quadratic in the size of the underlying algebra. All algorithms we develop find satisfying valuations if they exist.**

Keywords: Satisfiability, 2-CNF, multi-valued logics, regular sat, De Morgan algebra.

I. Introduction

Multi-valued logics occur naturally in computer science and artificial intelligence, especially when one wants to represent incomplete, partial or uncertain knowledge. Of particular interest in these applications is so-called signed logic. Signed SAT is NP-complete and therefore there is a natural challenge of identification its tractable subclasses and development fast algorithms for them [1], [4], [3], [5], [10].

In this paper, we investigate one of those polynomially solvable subclasses. In the first part, up to Section IV, we discuss the time complexity of 2-SAT with regular signs, where underlying truth value set forms a lattice. This problem is also known as regular 2-SAT with signs of the form $\uparrow i$ and $\downarrow i$ [4]. In the second part, we consider 2-SAT in logics over De Morgan algebras. As we show, this problem can be reduced to 2-SAT with regular signs. De Morgan algebra is a well known and deeply studied algebraic structure that still attracts a lot of attention of researchers. Recently, logics over these algebras were studied in the context of model checking [9], [8], [11], where they are used to reason about systems in early design stages.

Besides decision problems, we cope also with the corresponding functional ones. So we develop algorithms that not only answer whether a given formula is satisfiable or not, but also, in positive case, they give a satisfying valuation.

In [4], 2-SAT with regular signs was solved by an algorithm based on resolution. However, as we show in Section III, resolution does not give a satisfying valuation and therefore cannot come up with functional challenges.

Our contributions are algorithms that avoid this drawback and are faster than resolution. In particular, we solve 2-SAT with regular signs in quadratic time in the size of the input, and in the case with fixed lattice, in linear time in the size of a given formula. Our approach to these problems is based on reduction of a many-valued satisfiability problem to a classical one. Similar mappings were used before to develop algorithms for other tractable fragments of signed logic [5], [3].

To solve 2-SAT in logics over De Morgan algebras in a direct and efficient way, we use a very different method (sections VI,VII) and we obtain an algorithm that works in time $O(|\psi| + |L|^2)$ for a given formula ψ and a given truth value set L. What might be interesting is that a step in developing this algorithm is an algorithm for 2-SAT in Belnap's four-valued logic [6].

II. Preliminaries

A. Lattices

A *lattice* is a partial order $\mathcal{L} = (L, \sqsubseteq)$, where every finite subset $A \subseteq L$ has a least upper bound (called "join" and written $\sqcup A$) and a greatest lower bound (called "meet" and written $\sqcap A$). When a lattice is finite, it contains the greatest and the least elements denoted by $\top$ and $\bot$, respectively. A lattice is distributive if, for all $l_1, l_2, l_3 \in L$, we have: $l_1 \sqcap (l_2 \sqcup l_3) = (l_1 \sqcap l_2) \sqcup (l_1 \sqcap l_3)$, and $l_1 \sqcup (l_2 \sqcap l_3) = (l_1 \sqcup l_2) \sqcap (l_1 \sqcup l_3)$.

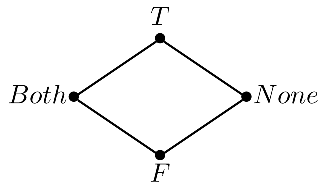

Fig. 1. The logical lattice L4 of Belnap's four-valued logic.

Definition 1: An element j in a lattice $\mathcal{L}$ is called *join irreducible* iff $j \neq \bot$ and for all $l_1, l_2 \in L$, $j = l_1 \sqcup l_2$ implies $j = l_1$ or $j = l_2$. The set of all join-irreducible elements in $\mathcal{L}$ is denoted by $\mathcal{J}(\mathcal{L})$.

For example, the set of join-irreducible elements of a lattice from Fig. 1 consists of elements $Both$ and $None$.

The order in distributive lattices distributes over join-irreducible elements: for all $j \in \mathcal{J}(\mathcal{L})$ we have $(l_1 \sqcup l_2) \sqsupseteq j$ iff $l_1 \sqsupseteq j$ or $l_2 \sqsupseteq j$, as well as $(l_1 \sqcap l_2) \sqsupseteq j$ iff $l_1 \sqsupseteq j$ and $l_2 \sqsupseteq j$.

Definition 2: Let $\mathcal{L} = (L, \sqsubseteq)$ be a lattice. A nonempty set $\mathcal{F} \subseteq L$ is called a *filter,* when $\mathcal{F} \neq L$ and

1) if $l_1, l_2 \in \mathcal{F}$, then $l_1 \sqcap l_2 \in \mathcal{F}$
2) if $l_1 \in \mathcal{F}$ and $l_2 \in L$, then $l_1 \sqcup l_2 \in \mathcal{F}$.

A filter $\mathcal{F}$ is said to be prime provided that if $l_1 \sqcup l_2 \in \mathcal{F}$, then either $l_1 \in \mathcal{F}$ or $l_2 \in \mathcal{F}$.

It is easy to see that every element $l \neq \bot$ in a lattice $\mathcal{L}$ generates a filter $\uparrow l = \{l_1 \in L \mid l_1 \sqsupseteq l\}$. Moreover, every filter in a lattice is generated by some element. Filters that corresponds to join-irreducibles are prime. Dually to a filter, we may define an ideal. But because we will not need its formal definition, we only note that all sets of the form $\downarrow l = \{l_1 \in L \mid l_1 \sqsubseteq l\}$ for some $l \neq \top$ are ideals and every ideal in a lattice may be seen as such a set.

Lemma 3: Let $(L, \sqsubseteq)$ be a lattice. Then for every $l_1, l_2 \in L$

$$\uparrow l_1 \cap \uparrow l_2 = \uparrow \sqcup \{l_1, l_2\} \tag{1}$$

$$\downarrow l_1 \cap \downarrow l_2 = \downarrow \sqcap \{l_1, l_2\} \tag{2}$$

$$\uparrow l_1 \cap \downarrow l_2 \neq \emptyset \textit{ iff } l_1 \sqsubseteq l_2 \tag{3}$$

Lemma 4: Let $l_1, \ldots l_m, l'_1, \ldots, l'_n$ be elements of a lattice $(L, \sqsubseteq)$ such that $l_i \sqsubseteq l'_j$ for all $i \leq m$ and $j \leq n$. Then

$$\bigcap_{i=1}^{m} \uparrow l_i \cap \bigcap_{j=1}^{n} \downarrow l'_j \neq \emptyset.$$

Proof: Since $l_1, \ldots, l_m$ are lower bounds of the set $\{l'_1, \ldots, l'_n\}$ and $l'_1, \ldots, l'_n$ are upper bounds of the set $\{l_1, \ldots, l_m\}$, we have $\sqcup\{l_1, \ldots, l_m\} \sqsubseteq \sqcap\{l'_1, \ldots, l'_n\}$. By (3) above, $\uparrow \sqcup\{l_1, \ldots, l_m\} \cap \downarrow \sqcap\{l'_1, \ldots, l'_n\} \neq \emptyset$. Finally, by (1) and (2) we obtain $\bigcap_{i=1}^{m} \uparrow l_i \cap \bigcap_{j=1}^{n} \downarrow l'_j \neq \emptyset$. ■

B. Syntax and semantics of Signed CNF formulas

1) Signed logic: We assume that a denumerable set $\Sigma = \{x, y, z, \ldots\}$ of propositional variables is given. Elements of Σ are interpreted in a universe called a *truth value set*. We restrict ourselves to the case, where truth value set L together with some relation $\sqsubseteq$ forms a finite lattice $\mathcal{L} = (L, \sqsubseteq)$.

Definition 5: A *sign* S is a subset of L. A *signed atom* is of the form $S : x$ where S is a sign and x is a propositional variable. A sign S is *regular* if it is identical to $\uparrow l$ or $\downarrow l$ for some $l \in L$.

Definition 6: A *signed clause* is a finite disjunction of signed atoms. A *signed CNF formula* Γ is a finite conjunction of signed clauses. A signed CNF formula whose clauses consists of at most two signed atoms is called a *signed 2-CNF formula*. A set of variables occurring in Γ is denoted by $Var(\Gamma)$.

As is usual, we identify a clause with the set of atoms it contains and a formula with the corresponding set of clauses.

Definition 7: The size of a signed clause C, denoted by $|C|$, is its cardinality. The size of a signed formula Γ, denoted by $|\Gamma|$, is the sum of the sizes of its signed clauses.

Definition 8: A *valuation* is a mapping that assigns to every propositional variable an element of the truth value set. A valuation ρ *satisfies* a signed atom $S : x$ iff $\rho(x) \in S$. It satisfies a signed clause iff it satisfies at least one of its atoms; it satisfies a signed CNF formula Γ if it satisfies all its clauses.

2) Satisfiability problems: Generally, the SAT problem for signed CNF formulas (*signed* SAT) is a question whether for a given truth value set L and a CNF formula Γ, there exists a valuation ρ such that ρ satisfies Γ.

Besides decision problems, it is reasonable to consider functional ones. An algorithm that solves a functional problem is required not only to answer: "yes" or "not", but also, in case of positive answer, to give a witness. Concerning satisfiability problems, a witness is a satisfying valuation. In this paper, we consider the following problems.

1) 2-SAT with regular signs — signed SAT restricted to 2-CNF formulas and signs of the form $\uparrow l$ and $\downarrow l$,
2) 2-SAT$_L$ with regular signs — a version of the problem above, where $(L, \sqsubseteq)$ is fixed,
3) 2-FSAT with regular signs — a functional version of 2-SAT with regular signs,
4) 2-FSAT$_L$ with regular signs — a functional version of the problem, with fixed $(L, \sqsubseteq)$.

III. Resolution and functional problems

In this section we give an example showing that resolution does not cope with functional problems.

As it was proved in [4], the resolution principle

$$\frac{\uparrow l_1 : x \vee D_1 \quad \downarrow l_2 : x \vee D_2}{D_1 \vee D_2} \quad \text{if } l_2 \not\sqsupseteq l_1$$

is refutation complete for 2-CNF formulas with regular signs, where underlying truth value set forms a lattice.

Now consider a truth value set from Fig. 1 and a formula Γ given as follows:

$$\begin{array}{lllll}
\uparrow Both : x & \vee & \uparrow Both : y, \uparrow None : x & \vee & \uparrow None : y, \\
\uparrow Both : x & \vee & \downarrow Both : y, \uparrow None : x & \vee & \downarrow None : y, \\
\downarrow Both : x & \vee & \uparrow Both : y, \downarrow None : x & \vee & \uparrow None : y. \\
\downarrow Both : x & \vee & \downarrow Both : y, \downarrow None : x & \vee & \downarrow None : y.
\end{array}$$

Using the resolution principle, we can obtain eight more clauses:

$$\begin{array}{lllll}
\uparrow Both : y & \vee & \uparrow None : y, \uparrow Both : y & \vee & \downarrow None : y, \\
\downarrow Both : y & \vee & \uparrow None : y, \downarrow Both : y & \vee & \downarrow None : y, \\
\uparrow Both : x & \vee & \uparrow None : x, \uparrow Both : x & \vee & \downarrow None : x, \\
\downarrow Both : x & \vee & \uparrow None : x, \downarrow Both : x & \vee & \downarrow None : x.
\end{array}$$

Define $\rho_1(x) = Both$, $\rho_1(y) = None$ and $\rho_2(x) = None$, $\rho_2(y) = Both$. It is straightforward to check that both these valuations satisfy Γ. However, since Γ is satisfiable, no empty clause can be derived with this resolution principle. Moreover, since the two satisfying valuations assign different values to the same variables, no unit clause can be obtained. Therefore the resolution does not say anything about satisfying valuation and is not proper to solve functional problems.

IV. Modular Reductions of 2-SAT Problems with Regular Signs to Classical Logic

In this section, we reduce SAT problems introduced in Section II-B to classical SAT. Now, we show a way of how to transform a CNF formula Γ with regular signs into a classical CNF formula Γ^δ. This reduction will be called δ.

Let Σ be a signature of Γ, then Σ^δ, the signature of Γ^δ, is $\{v_{\Uparrow l:x} \mid l \in L,\ x \in \Sigma\} \cup \{v_{\downarrow l:x} \mid l \in L,\ x \in \Sigma\}$. The reduction δ follows in two steps. First, for every clause $(S_1 : x \vee S_2 : y)$ over Σ of Γ, formula Γ^δ contains a corresponding clause $(v_{S_1:x} \vee v_{S_2:y})$ over signature Σ^δ. Then for every variable x of Σ, and for every $l_1, l_2 \in L$ such that $l_1 \not\sqsubseteq l_2$, we have a clause $(\neg v_{\Uparrow l_1:x} \vee \neg v_{\Downarrow l_2:x})$ in Γ^δ. The set of these extra clauses will be denoted by Γ_2^δ.

Now observe that $\left|\Gamma^\delta\right| \leq |\Gamma|\,|L|^2$ and that Γ^δ may be computed in time $O(|\Gamma|\,|L|^2)$.

Theorem 9: Let Γ be a CNF-formula with regular signs. Let Γ^δ be a classical CNF formula derived from Γ via reduction δ. Then, Γ is satisfiable iff Γ^δ is satisfiable.

Theorem 10: The problem 2-FSAT$_L$ with regular signs may be solved in time $O(|\Gamma|)$.

Using Theorem 9 and observations preceding it, 2-SAT and 2-FSAT with regular signs may be solved in time $O(|\Gamma|\,|L|^2)$. By restricting Γ_2^δ to contain only atoms that occur in Γ we get that 2-FSAT with regular signs is solvable in quadratic time in the size of the input.

Theorem 11: The problem 2-FSAT with regular signs may be solved in time $O(|\Gamma|^2 + |L|\,|\Gamma|)$.

Proofs of theorems 9–11 can be found in the appendix.

V. De Morgan Algebras

A. Preliminaries

Definition 12: A De Morgan Algebra $\mathcal{D}$ is a tuple $(\mathcal{L}, \sqcup, \sqcap, \neg)$, where $\mathcal{L} = (L, \sqsubseteq)$ is a distributive finite lattice and the function $\neg$ is any operation preserving the following properties.

- De Morgan laws

$$\neg(l_1 \sqcap l_2) = \neg l_1 \sqcup \neg l_2 \tag{4}$$

$$\neg(l_1 \sqcup l_2) = \neg l_1 \sqcap \neg l_2 \tag{5}$$

- involution

$$\neg(\neg l_1) = l_1 \tag{6}$$

Note that $\neg\top = \bot$ and $\neg\bot = \top$, but not necessarily $l \sqcap \neg l = \bot$ and $l \sqcup \neg l = \top$. The De Morgan laws imply another property that we use, namely *antimonotonicity*: for every two elements l_1, l_2 in any De Morgan algebra we have

$$l_1 \sqsubseteq l_2 \text{ if and only if } \neg l_2 \sqsubseteq \neg l_1 \tag{7}$$

As examples of De Morgan algebras consider a simple two-valued algebra corresponding to the classical logic and the lattice L4 from Fig. 1 augmented with negation defined as follows: $\neg T = F$, $\neg Both = Both$, and $\neg None = None$. The latter defines the truth value set of Belnap's logic. For the sake of simplicity, we will call this algebra Belnap's algebra ($\mathcal{B}$).

B. Characterization Theorem

Here, we define another useful example of De Morgan algebra, and we show its properties. The notation $\mathcal{P}(U)$ stands for the family of all subsets of the set U.

Definition 13: Let U be a nonempty set and let $\mathbf{neg} : U \to U$ be an involution. A *quasi-field of sets* is a tuple $(Q(U), \cap, \cup, \neg)$, where $Q(U)$ is a subfamily of $\mathcal{P}(U)$ closed under $\cap$, $\cup$ and under the operation $\neg$ called quasi-complement defined by

$$\neg X = U \setminus \{\mathbf{neg}(u) \mid u \in X\}$$

for any $X \in Q(U)$.

Observe that every quasi-field of sets is a De Morgan algebra. It turns out that the reverse is also true: every De Morgan algebra may be seen as a quasi-field of sets. The following characterization theorem originates from [7].

Theorem 14 (Białynicki-Birula, Rasiowa): Every De Morgan Algebra $(\mathcal{L}, \sqcup, \sqcap, \neg)$ is isomorphic to a quasi-field of sets $(Q(\mathcal{Y}), \cap, \cup, \neg)$, where $\mathcal{Y}$ is a set of all prime filters of $\mathcal{L}$. The isomorphism $\mathcal{I} : \mathcal{L} \to Q(\mathcal{Y})$ may be defined by $\mathcal{I}(l) = \{\mathcal{F} \in \mathcal{Y} \mid l \in \mathcal{F}\}$.

Because of the observation following Definition 2, Theorem 14 may be reformulated as follows.

Theorem 15: Every De Morgan Algebra $(\mathcal{L}, \sqcup, \sqcap, \neg)$ is isomorphic to a quasi-field of sets $(Q(\mathcal{J}(\mathcal{L})), \cap, \cup, \neg)$. The isomorphism $\mathcal{I} : \mathcal{L} \to Q(\mathcal{J}(\mathcal{L}))$ may be defined by $\mathcal{I}(l) = \{j \in \mathcal{J}(\mathcal{L}) \mid j \sqsubseteq l\}$.

Because it is more convenient to work with elements (join-irreducibles) rather than with sets (filters), in the following we will use the characterization from Theorem 15.

C. Syntax and Semantics of Logics over De Morgan Algebras

As in the case of signed logic, we will use here the set of propositional variables Σ, operators $\vee$ and $\wedge$, and valuation ρ. Although it might lead to confusion, from the context it will always be clear whether we work with signed logic or with logics over De Morgan algebras.

Definition 16: A *literal* is a variable from Σ or its negation. A *clause* is an expression of the form $\alpha_1 \vee \ldots \vee \alpha_n$, where all α_i are literals. A *CNF formula* ψ is of the form $\bigwedge_{i=i}^n C_i$, where all C_i are clauses. A *2-CNF formula* is a CNF formula with at most two literals per clause. A set of variables occurring in ψ is denoted by $Var(\psi)$.

Definition 17: Let $\mathcal{D} = ((L, \sqsubseteq), \sqcup, \sqcap, \neg)$ be some De Morgan algebra. A *valuation of variables* in L is a mapping $\rho : \Sigma \to L$. An *interpretation* $\|\cdot\|_\rho$ of CNF formulas over the valuation ρ is given as follows:

- $\|x\|_\rho = \rho(x)$
- $\|\neg x\|_\rho = \neg\rho(x)$
- $\|\alpha_1 \vee \ldots \vee \alpha_n\|_\rho = \|\alpha_1\|_\rho \sqcup \ldots \sqcup \|\alpha_n\|_\rho$
- $\|C_1 \wedge \ldots \wedge C_m\|_\rho = \|C_1\|_\rho \sqcap \ldots \sqcap \|C_m\|_\rho$

Note that every formula satisfiable in classical propositional logic is satisfied in every De Morgan algebra by $\top$ and $\bot$. But there are De Morgan algebras such that for some of their elements, classically unsatisfiable formulas are satisfied. For example, let us consider Belnap's algebra, element $Both$ and the formula $x \wedge \neg x$. It is easily seen that $Both \sqcap \neg Both \sqsupseteq Both$.

D. Satisfiability Problem

Here, we define a 2-SAT problem in De Morgan algebras and show that it reduces to 2-SAT with regular signs.

Definition 18: The 2-SAT problem in logics over De Morgan algebras depends on three parameters: a 2-CNF formula

ψ, a De Morgan algebra $\mathcal{D} = ((L, \sqsubseteq), \sqcup, \sqcap, \neg)$ and an element $l \neq \bot$ of L. We ask whether there exists a valuation $\rho : \Sigma \to L$ such that $\|\varphi\|_\rho \sqsupseteq l$. If it is the case, then we say that ρ satisfies ψ for l. The 2-FSAT problem in logics over De Morgan algebras is a functional version of this problem.

In context of multi-valued satisfiability, the set of *designated truth values* is often defined. Here we note that for an instance of 2-SAT problem, namely $(\psi, \mathcal{D}, l)$, the set of designated truth values is equal to $\{l_1 | l_1 \sqsupseteq l\}$.

Theorem 19: For every instance $(\psi, \mathcal{D}, l)$ of 2-SAT problem in logic over De Morgan algebras there exists a signed formula Γ over L, of size $O(|\psi||L|)$, with regular signs, such that ψ is satisfiable in $\mathcal{D}$ for l iff Γ is satisfiable in L.

Proof: Let $\psi = \bigwedge_{i=1}^{n} (\alpha_i \vee \beta_i)$. Let $J = \{j \in \mathcal{J}(\mathcal{L}) \mid j \sqsubseteq l\}$. Then $l = \bigsqcup_{j \in J} j$ and since the order distributes over join-irreducible elements, we have $\|\psi\|_\rho \sqsupseteq l$, for some ρ, iff for all $i = 1, \dots n$ and for all $j \in J$ we have $\|\alpha_i\|_\rho \sqsupseteq j$ or $\|\beta_i\|_\rho \sqsupseteq j$. For negative literals of the form $\neg x$ for some variable x, due to the antimonotonicity of negation, we have $\|\neg x\|_\rho \sqsupseteq j$ iff $\|x\|_\rho \sqsubseteq \neg j$. Therefore $\|x\|_\rho \sqsupseteq j$ iff ρ satisfies a signed atom $\uparrow j : x$ and $\|\neg x\| \sqsupseteq j$ iff ρ satisfies a signed atom $\downarrow (\neg j) : x$. Now it is enough to take Γ as the signed formula

$$\Gamma = \bigwedge_{j \in J} \bigwedge_{i=1}^{n} (S(\alpha_i, j) \vee S(\beta_i, j))$$

where $S(x, j) = \uparrow j : x$ and $S(\neg x, j) = \downarrow (\neg j) : x$. ■

Example 20: Consider a formula $\psi = (x \vee y) \wedge (x \vee \neg y) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg y)$. It is easy to check that reduction from Theorem 19 transforms ψ to signed formula Γ from Section III.

The theorem above together with the observation following Theorem 10 gives for any De Morgan algebra a method for solving 2-SAT and 2-FSAT problems in time $O(|\psi||L|^3)$. In the rest of the paper we show that this can be improved to $O(|\psi| + |L|^2)$.

VI. 2-FSAT PROBLEM IN BELNAP'S LOGIC

In this section, we discuss conditions on 2-CNF formulas to be satisfiable in Belnap's algebra. First, in Section VI-A we examine when a 2-CNF formula is satisfiable for T, and then in Section VI-B we consider the problem for *Both* and *None*. Later, in Section VII we will show how to extend the method to arbitrary De Morgan algebras.

Below, we identify any variable x with its double negation $\neg\neg x$. The notation $Literals(\psi)$ is used for the set $Var(\psi) \cup \{\neg x \mid x \in Var(\psi)\}$. Moreover, we use $\rho_{cl} : \Sigma \to \{0, 1\}$ to be a classical valuation, and $\rho_{\mathcal{B}} : \Sigma \to \{T, Both, None, F\}$ to be a valuation in carrier set of Belnap's algebra.

A. Valuations satisfying formulas for T

Definition 21: Let ψ be an arbitrary 2-CNF formula. The satisfiability graph for ψ is a directed graph $G^S_\psi = (V^S_\psi, E^S_\psi)$ where the set of vertices V^S_ψ consists of $Literals(\psi)$ and for every clause $(\alpha \vee \beta)$ of ψ there is an edge $(\neg\alpha, \beta), (\neg\beta, \alpha) \in E^S_\psi$.

Let us shortly remind the algorithm for classical 2-SAT [2]. First, for a given formula ψ, we construct G^S_ψ. Then the formula ψ is satisfiable iff there is no cycle containing x and $\neg x$, for any variable x.

If G'^S_ψ is a maximal strongly connected component (MSCC) of G^S_ψ that contains x and $\neg x$, then we will call it a *classically unsatisfiable component*.

Lemma 22: Let ψ be a 2-CNF formula, G^S_ψ its satisfiability graph and $G'^S_\psi = (V'^S_\psi, E'^S_\psi)$ a classically unsatisfiable component of G^S_ψ. If $\|\psi\|_{\rho_{\mathcal{B}}} \sqsupseteq T$, then for every $x \in G'^S_\psi$, we have $\rho_{\mathcal{B}}(x) \in \{Both, None\}$.

Proof: Assume on the contrary that ψ is classically unsatisfiable and that there is a valuation $\rho_{\mathcal{B}}$ satisfying ψ for T that maps at least one variable $y \in V'^S_\psi$ to T or F. Then the maximal strongly connected component G'^S_ψ contains a cycle (subformula) of the form: $(\neg\alpha \vee \beta_1) \wedge (\neg\beta_1 \vee \beta_2) \wedge \dots \wedge (\neg\beta_m \vee \neg\alpha) \wedge (\alpha \vee \gamma_1) \wedge \dots \wedge (\neg\gamma_n \vee \alpha)$. Moreover, our supposed variable y is some α, β, $\neg\alpha$ or γ of at least one of such cycles in G'^S_ψ. One can show by a simple induction that if at least one of the literals $\alpha, \beta_i, \neg\alpha, \gamma_j$ is evaluated to T or F, then all these literals (including α and $\neg\alpha$) must be evaluated to the same element of $\mathcal{B}$ and in consequence ψ cannot be satisfied for T. ■

Definition 23: Let ψ be an arbitrary 2-CNF formula. Then a colour graph G corresponding to an MSCC G'^S_ψ of G^S_ψ is an undirected graph (V, E) such that $V = Var(\psi) \cap V'^S_\psi$ and for every $x, y \in Var(\psi)$, there is an edge $\{x, y\} \in E$ iff E'^S_ψ contains an edge of the form (x, y), $(\neg x, y)$, $(x, \neg y)$ or $(\neg x, \neg y)$.

Lemma 24: Let ψ be a 2-CNF formula satisfiable for T. Then for every classically unsatisfiable MSCC G'^S_ψ of G^S_ψ, the corresponding colour graph is 2-colourable.

Proof: Suppose that $\rho_{\mathcal{B}}$ satisfies ψ for T. Take any classically unsatisfiable MSCC of G^S_ψ and let G be the corresponding colour graph. By Lemma 22, we know that for all variables x occurring in G, we have $\rho_{\mathcal{B}}(x) \in \{Both, None\}$. Define a colouring $\mathcal{C}$ of the graph such that $\mathcal{C}(x) = 1$ iff $\rho_{\mathcal{B}}(x) = Both$, and $\mathcal{C}(x) = 2$ iff $\rho_{\mathcal{B}}(x) = None$. This is a 2-colouring of G'^C_ψ — otherwise, if there is an edge whose both ends x and y are coloured with the same colour k, then ψ contains a clause that evaluates to $Both$ or $None$ under $\rho_{\mathcal{B}}$, which contradicts the assumption that $\rho_{\mathcal{B}}$ satisfies ψ for T. ■

Lemma 24 gives a necessary condition for 2-CNF formulas to be satisfied for T. In the following part of this section, we show that this condition is also sufficient. But first, let us give some lemmas about the structure of satisfiability graphs.

For $\alpha, \beta \in V^S_\psi$, we write $\alpha \prec_S \beta$ iff there exists a path from α to β in G^S_ψ. We extend $\prec_S$ to MSCCs of G^S_ψ in the following way. Let G'^S_ψ, G''^S_ψ be some MSCCs of G^S_ψ. Then we define $G'^S_\psi \prec_S G''^S_\psi$ iff there exist $\alpha \prec_S \beta$ such that $\alpha \in G'^S_\psi$ and $\beta \in G''^S_\psi$. Similarly, we write $\alpha \prec_S G'^S_\psi$ if there exists $\beta \in G'^S_\psi$ such that $\alpha \prec_S \beta$.

Lemma 25: Let $\alpha, \beta \in V^S_\psi$. If $\alpha \prec_S \beta$ then $\neg\beta \prec_S \neg\alpha$.

Proof: Proof may be done by a simple induction, using Definition 21 in the base case. ■

Observe that Lemma 25 implies that if a literal α occurs in some classically unsatisfiable MSCC then its negation $\neg\alpha$ must occur there, too.

Lemma 26: Let G'^S_ψ, G''^S_ψ be classically unsatisfiable MSCCs of G^S_ψ. If $G'^S_\psi \prec_S G''^S_\psi$, then $G'^S_\psi = G''^S_\psi$.

Proof: The proof can be found in the appendix. ■

The above lemma implies that all classically unsatisfiable MSCCs of G^S_ψ are incomparable in sense of $\prec_S$.

Lemma 27: Let ψ be a 2-CNF formula such that for every classically unsatisfiable MSCC G'^S_ψ of G^S_ψ, the corresponding colour graph is 2-colourable. Then $\|\psi\|_{\rho_\mathcal{B}} \sqsupseteq T$ for some valuation $\rho_\mathcal{B}$.

Proof: For each classically unsatisfiable component G'^S_ψ of G^S_ψ, we introduce a new variable $v_{G'^S_\psi}$ and replace in the graph G^S_ψ the component G'^S_ψ with a single edge $(\neg v_{G'^S_\psi}, v_{G'^S_\psi})$ (where each edge coming from outside to G'^S_ψ is replaced by an edge leading to $\neg v_{G'^S_\psi}$ and each edge going from G'^S_ψ to outside is replaced by an edge originating in $v_{G'^S_\psi}$). The new graph is a satisfiability graph G^S_φ of a new formula φ obtained from ψ by

- removing all clauses $\alpha \vee \beta$ where both α and β are in the same classically unsatisfiable MSCC,
- adding clauses $v_{G'^S_\psi} \vee v_{G'^S_\psi}$ for all classically unsatisfiable MSCC G'^S_ψ,
- replacing all clauses of the form $\beta \vee \alpha$ with $\beta \vee \neg v_{G'^S_\psi}$ where $\alpha \in V'^S_\psi$ and $\beta \notin V'^S_\psi$.

The formula φ is classically satisfiable — otherwise the graph G^S_φ must contain a classically unsatisfiable strongly connected component, which contradicts the fact that we already removed all such components.

Let ρ_{cl} satisfy φ, and let $\mathcal{C}$ be a 2-colouring of all colour graphs of classically unsatisfiable MSCCs of G^S_ψ. For all variables x occurring in some classically unsatisfiable MSCC define $\rho_\mathcal{B}(x) = Both$ iff $\mathcal{C}(x) = 1$ and $\rho_\mathcal{B}(x) = None$ iff $\mathcal{C}(x) = 2$. For all remaining variables define $\rho_\mathcal{B}(x) = T$, whether $\rho_{cl}(x) = 1$ and $\rho_\mathcal{B}(x) = F$, whether $\rho_{cl}(x) = 0$. Note that by (the observation following) Lemma 25 the two sets of variables (corresponding to the colouring $\mathcal{C}$ and to the assignment ρ_{cl}) are disjoint, so the definition of $\rho_\mathcal{B}$ is correct.

We claim that $\rho_\mathcal{B}$ is a satisfying assignment for ψ. To see this, observe that

- all clauses $\alpha \vee \beta$, where both α and β are in the same classically unsatisfiable MSCC, are satisfied: due to 2-colourability of the corresponding colour graph, $\|\cdot\|_\mathcal{B}$ assigns different values $Both$ and $None$ to α and β, and their join evaluates to T,
- ρ_{cl} assigns T to all variables $v_{G'^S_\psi}$,
- all clauses of the form $\beta \vee \alpha$, where $\alpha \in V'^S_\psi$ and $\beta \notin V'^S_\psi$, evaluate to T since $\|\beta\|_{\rho_{cl}} = 1$,
- by Lemma 26, there are no clauses $\alpha \vee \beta$ with α and β occurring in different classically unsatisfiable MSCC.

■

The following proposition is a direct consequence of lemmas 24 and 27.

Proposition 28: Let ψ be a classically unsatisfiable 2-CNF formula. Then there exists $\rho_\mathcal{B}$ such that $\|\psi\|_{\rho_\mathcal{B}} \sqsupseteq T$ iff for every classically unsatisfiable MSCC of G^S_ψ, its corresponding colour graph is 2-colourable.

Example 29: Consider $\varphi = \psi \wedge (\neg u \vee x) \wedge (\neg u \vee y)$, where ψ is a formula defined in Example 20 in Section V. Let us show a valuation $\rho_\mathcal{B}$, from the proof of Lemma 27, that satisfies φ for T. The satisfiability graph G^S_φ has three MSCCs. The first one contains only one node u, the second contains nodes $x, y, \neg x, \neg y$ and the third one node $\neg u$. Define $\mathcal{C}$ to be a colouring of colour graph corresponding to the middle MSCC above such that $\mathcal{C}(x) = 1$ and $\mathcal{C}(y) = 2$. It is not hard to check that $\mathcal{C}$ is really a 2-colouring. Further, $\rho_\mathcal{B}$ evaluates x, y according to $\mathcal{C}$. Then we replace this MSCC with an edge $(\neg v, v)$, obtaining a graph with three edges: $(u, \neg v), (\neg v, v)$ and $(v, \neg u)$. The classical formula corresponding to this new graph is $(\neg u \vee \neg v) \wedge (v \vee v)$, so a satisfying valuation must evaluate u to 0. Hence, $\rho_\mathcal{B}$ such that $\rho_\mathcal{B}(x) = Both$, $\rho_\mathcal{B}(y) = None$, and $\rho_\mathcal{B}(u) = F$ satisfies φ for T.

B. Valuation satisfying formulas for $Both$ and $None$.

First note that for elements $Both$ and $None$, all formulas are satisfiable. It is enough to evaluate every variable to $Both$ or $None$, respectively. Nevertheless, we need to know one simple fact more about satisfying valuations for these truth values.

Lemma 30: Let ψ be a classically unsatisfiable 2-CNF formula. Let $W \in \{Both, None\}$, and $\|\psi\|_{\rho_\mathcal{B}} \sqsupseteq W$. Then for every clause $\alpha \vee \beta$, where α, β belong to some classically unsatisfiable component of G^S_ψ, we have $\|\alpha\|_{\rho_\mathcal{B}} = W$ or $\|\beta\|_{\rho_\mathcal{B}} = W$.

Proof: We prove the proposition for $W = Both$, the other case is symmetric. If both α, β are different from $Both$ and $\|\alpha \vee \beta\|_{\rho_\mathcal{B}} \sqsupseteq Both$, then $\|\alpha \vee \beta\|_{\rho_\mathcal{B}} = T$. By Lemma 22, we have $\|\alpha\|_{\rho_\mathcal{B}}, \|\beta\|_{\rho_\mathcal{B}} \in \{Both, None\}$ and in consequence $\|\alpha \vee \beta\|_{\rho_\mathcal{B}} = None$, which contradicts the assumption that $\|\psi\|_{\rho_\mathcal{B}} \sqsupseteq Both$. ■

VII. 2-FSAT Problem in Logics over De Morgan Algebras

Let $\mathcal{D} = ((L, \sqsubseteq), \sqcup, \sqcap, \neg)$ be a De Morgan Algebra, $l \neq \bot$ an element of its carrier set, ψ a 2-CNF formula, and ρ a valuation of $Var(\psi)$ in L. Further, an isomorphism from Theorem 15 for $\mathcal{D}$ is denoted by $\mathcal{I}$ and for Belnap's logic by $\mathcal{I}_B$.

A. Reduction

Here, we reduce 2-SAT for logics over De Morgan algebras to 2-SAT for Belnap's logic and 2-SAT for classical logic. To understand it better note that classical logic is the simplest De Morgan algebra that contains some join-irreducible element j such that $\mathbf{neg}(j) = j$: in classical logic we have $\mathbf{neg}(1) = 1$. Similarly, Belnap's logic is the simplest one that contains a join-irreducible element j such that $\mathbf{neg}(j) \neq j$, we have $\mathbf{neg}(Both) = None$.

Note that an isomorphic image $\mathcal{I}(l)$ of $l \in \mathcal{L}$ may be split into three different sets: $J^l_1 = \{j \in \mathcal{J}(\mathcal{L}) \mid \mathbf{neg}(j) = j\}$,

$J_2^l = \{j \in \mathcal{J}(\mathcal{L}) \mid \mathbf{neg}(j) \neq j \ and \ \mathbf{neg}(j) \in \mathcal{I}(l)\}$, and $J_3^l = \{j \in \mathcal{J}(\mathcal{L}) \mid \mathbf{neg}(j) \notin \mathcal{I}(l)\}$.

Now, provided that $J_1^l \neq \emptyset$, we reduce 2-SAT in logics over De Morgan algebras to classical 2-SAT. The following lemma implies that if J_1^l is not empty then every formula satisfiable for l is classically satisfiable.

Lemma 31: Let $\|\psi\|_\rho \sqsupseteq l$, and $i \in J_1^l$. Define $\rho_{cl}(x) = 1$ iff $i \in \rho(x)$, and $\rho_{cl}(x) = 0$ iff $i \notin \rho(x)$. Then $\|\psi\|_{\rho_{cl}} \sqsupseteq 1$.

Proof: Because $\mathbf{neg}(i) = i$, by Definition 13 and Theorem 15, it can be easily checked that $i \in \mathcal{I}(\|\neg x\|_\rho)$ iff $\|\neg x\|_{\rho_{cl}} = 1$.

Seeking for contradiction assume that $\|\psi\|_\rho \sqsupseteq l$ and ρ_{cl} does not satisfy ψ in classical logic. Then ψ contains a clause $(\alpha \vee \beta)$ such that $\|\alpha\|_{\rho_{cl}} \neq 1$ and $\|\beta\|_{\rho_{cl}} \neq 1$. Further, from the definition of ρ_{cl} follows that $i \notin \mathcal{I}(\|(\alpha \vee \beta)\|_\rho)$. Therefore ρ does not satisfy ψ for i; and since $i \sqsubseteq l$, ρ does not satisfy ψ for l. Thus we obtained the contradiction and proved the lemma. ∎

The next lemma shows that in the case of $J_1^l = \emptyset$ and $J_2^l = \emptyset$, the problem becomes trivial.

Lemma 32: Let $J_1^l = \emptyset$, and $J_2^l = \emptyset$. For every variable x define $\rho(x) = l$. Then $\|\psi\|_\rho \sqsupseteq l$ for any CNF formula ψ.

Proof: From Definition 13 and Theorem 15 follows that $\mathcal{I}(\neg l) \supseteq \mathcal{I}(l)$ and hence $\neg l \sqsupseteq l$. So ρ satisfies ψ for l. ∎

Finally, for $J_1^l = \emptyset$ and $J_2^l \neq \emptyset$, we reduce 2-SAT in logics over De Morgan algebras to 2-SAT in Belnap's logic.

Theorem 33: Let $J_1^l = \emptyset$, and $J_2^l \neq \emptyset$. Let ψ be a classically unsatisfiable 2-CNF formula. Then, there exists ρ satisfying ψ for l iff all of the following holds.

1) There exist elements $\mathbf{l_1}, \mathbf{l_2} \in L$ such that:
   a) for every $i \in J_2^l$, we have $i \in \mathcal{I}(\mathbf{l_1}) \setminus \mathcal{I}(\mathbf{l_2})$, and $\mathbf{neg}(i) \in \mathcal{I}(\mathbf{l_2}) \setminus \mathcal{I}(\mathbf{l_1})$ or $\mathbf{neg}(i) \in \mathcal{I}(\mathbf{l_1}) \setminus \mathcal{I}(\mathbf{l_2})$, and $i \in \mathcal{I}(\mathbf{l_2}) \setminus \mathcal{I}(\mathbf{l_1})$;
   b) for every $i \in J_3^l$, we have $i \in \mathcal{I}(\mathbf{l_1})$, and $\mathbf{neg}(i) \notin \mathcal{I}(\mathbf{l_1})$ or $i \in \mathcal{I}(\mathbf{l_2})$, and $\mathbf{neg}(i) \notin \mathcal{I}(\mathbf{l_2})$.
2) There exists valuation $\rho_\mathcal{B}$ satisfying ψ for T.

Proof: Proofs of the theorem and related lemmas can be found in the appendix. ∎

B. Algorithm for 2-FSAT for an arbitrary De Morgan Algebra.

Here, we show an algorithm for 2-FSAT for logics over De Morgan algebras.

Algorithm 34: Let $\mathcal{D} = ((L, \sqsubseteq), \sqcup, \sqcap, \neg)$ be an arbitrary De Morgan algebra, $l \neq \bot$ an element of its carrier set, and ψ a 2-CNF formula. If there exists ρ such that $\|\psi\|_\rho \sqsupseteq l$, then the algorithm returns ρ, otherwise it returns *no*.

1) Find the set of MSCCs of G_ψ^S.
2) If there is no classically unsatisfiable component, then go to point 12.
3) Compute $\mathcal{J}(\mathcal{L})$.
4) Find the isomorphism $\mathcal{I}$.
5) Compute the function $\mathbf{neg}$.
6) Find J_1^l, J_2^l, and J_3^l.
7) If $J_1^l \neq \emptyset$, then return *no*.
8) If $J_2^l = \emptyset$, then for every $x \in V_\psi^S$ define $\rho(x) = l$. Go to point 14.
9) If there are no elements $\mathbf{l_1}, \mathbf{l_2} \in L$ as required in Theorem 33, then return *no*.
10) If there is any classically unsatisfiable $G_\psi'^S$ such that its corresponding colour graph is not 2-colourable, then return *no*.
11) Let $G_\psi'^S$ be some classically unsatisfiable components and let $\mathcal{C} : Var(\psi) \to \{1, 2\}$ be some 2-colouring of its corresponding colour graph. Then for every $x \in V_\psi'^S$, for $k = 1, 2$ define $\rho(x) = l_k$, whether $\mathcal{C}(x) = k$.
12) Replace every unsatisfiable component $G_\psi'^S$ of G_ψ^S with an edge $(\neg v_{G_\psi'^S}, v_{G_\psi'^S})$.
13) Find classical valuation ρ_{cl} for a formula induced by a new graph. For every variable x distinct from $v_{G_\psi'^S}$ define $\rho(x) = \top$, whether $\rho_{cl}(x) = 1$ and $\rho(x) = \bot$, whether $\rho_{cl}(x) = 0$.
14) Return ρ.

Theorem 35: Algorithm 34 is sound and complete.

Theorem 36: Algorithm 34 works in time $O(|\psi| + |L|^2)$.

Proofs of theorems 35 and 36 can be found in the appendix.

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APPENDIX

A. Proofs of Theorems 9, 10, and 11

In the following theorems let $\Gamma_1^\delta = \Gamma^\delta \setminus \Gamma_2^\delta$.

Restatement of Theorem 9: Let Γ be a CNF-formula with regular signs. Let Γ^δ be a classical CNF formula derived from Γ via reduction δ. Then, Γ is satisfiable iff Γ^δ is satisfiable.

Proof: First assume that a valuation ρ satisfies Γ.

Let $\rho^\delta : \Sigma^\delta \to \{0,1\}$ be a classical valuation such that $\rho^\delta(v_{\Uparrow l:x}) = 1$ iff $\rho(x) \sqsupseteq l$, and $\rho^\delta(v_{\downarrow l:x}) = 1$ iff $\rho(x) \sqsubseteq l$. It is easily seen that ρ^δ satisfies Γ_1^δ. Every clause of Γ_2^δ is of the form: $(\neg v_{\Uparrow l_1:x} \vee \neg v_{\Downarrow l_2:x})$, where $\uparrow l_1 \cap \downarrow l_2 = \emptyset$. Since $\rho(x)$ cannot belong to two disjoint sets, we have that ρ^δ satisfies Γ_2^δ and hence Γ^δ.

Now assume a classical valuation ρ^δ to satisfy Γ^δ.

First, observe that if $\rho^\delta(v_{\Uparrow l:x}) = 1$ and $\rho^\delta(v_{\downarrow l':x}) = 1$, then $l \sqsubseteq l'$ — otherwise, by Lemma 3.(3), ρ^δ would not satisfy the clause $\neg v_{\Uparrow l:x} \vee \neg v_{\Downarrow l':x}$ in Γ_2^δ.

Let $x \in \Sigma$ and let $S_1, \ldots, S_n$ be all signs occuring in Γ such that $\rho^\delta(v_{S_k:x}) = 1$ for $1 \leq k \leq n$. By Lemma 4, the intersection $\bigcap_{k=1}^n S_k$ is not empty. Therefore, for every variable $x \in \Sigma$ we may define $\rho(x)$ to be any element of this intersection. It is now easy to observe that independently on l, valuation ρ satisfies Γ: In every clause of Γ_1^δ there is at least one member of Σ^δ such that ρ^δ satisfies it. By definition of ρ and by construction of Γ_1^δ, we obtain that ρ satisfies Γ. ■

Restatement of Theorem 10: The problem 2-FSAT$_L$ with regular signs may be solved in time $O(|\Gamma|)$.

Proof: First reduce Γ to Γ^δ via reduction δ and then use a well-known algorithm for 2-SAT [2] to check whether Γ^δ is satisfiable and to find a satisfying valuation ρ^δ. (By Theorem 9, we know that Γ and Γ^δ are equisatisfiable.) Then by observations preceding Theorem 9, we obtain that 2-SAT$_L$ with regular signs may be done in linear time.

Now we show how to find a valuation ρ satisfying Γ. Let x occur in Γ. Define $\mathcal{S}_x = \{\uparrow l_1, \ldots, \uparrow l_m, \downarrow l'_1, \ldots, \downarrow l'_n\}$ to be a set of signs such that $S \in \mathcal{S}_x$ iff $\rho^\delta(S : x) = 1$. From the proof of Theorem 9 follows that $\rho(x)$ may be any element of $\bigcap_{i=1}^m \uparrow l_i \cap \bigcap_{j=1}^n \downarrow l_j$. Therefore by Lemma 4, it is enough to choose $\uparrow \sqcup \{l_1, \ldots, l_m\}$ for $\rho(x)$.

Note that function $\sqcup$ is not given on the input and therefore we must compute that part of it that we need. In fact we are looking for $min(\{l \in L | \forall l' \in S_x l \sqsupseteq l'\})$, so it may be done in time $|S_x|\,|\mathcal{L}|$. Since $\sum_{x \in Var(\Gamma)} |S_x| \leq |\Gamma|$, the valuation ρ may be found in time $O(|\Gamma|\,|L|)$. ■

Let δ' be a version of δ such that $\Gamma_1^{\delta'} = \Gamma_1^\delta$ and $\Gamma_2^{\delta'}$ is Γ_2^δ restricted to contain only atoms that occur in Γ.

Restatement of Theorem 11: The problem 2-FSAT with regular signs may be solved in time $O(|\Gamma|^2 + |L|\,|\Gamma|)$.

Proof: The proof of the fact that Γ and $\Gamma^{\delta'}$ are equisatisfiable is similar to the proof of Theorem 9.

Now, we show how to compute $\Gamma_2^{\delta'}$ in time $O(|\Gamma|^2)$. For every variable x construct an array A_x. Rows of A_x index by signed atoms of the form $\uparrow l_1 : x$ and columns by $\downarrow l_2 : x$; use only signed atoms that occur in Γ. Next fill A_x with 0's and 1's in order to obtain $A_x(\uparrow l_1 : x, \downarrow l_2 : x) = 1$ iff $l_1 \not\sqsubseteq l_2$. Then, $\Gamma^{\delta'}$ contains a clause $(\neg v_{\Uparrow l_1:x} \vee \neg v_{\Downarrow l_2:x})$ iff $A_x(\uparrow l_1 : x, \downarrow l_2 : x) = 1$. Since $\Sigma_{x \in Var} |A_x| \leq |\Gamma|^2$ and each A_x may be computed in time linear in its size, we can obtain $\Gamma_2^{\delta'}$ in time $O(|\Gamma|^2)$

Next, as in the proof of Theorem 10 we use the algorithm for classical 2-SAT to deicide whether $\Gamma^{\delta'}$ is satisfiable and, in case of positive answer, to find a satisfying valuation. ■

B. Proof of Lemma 26

Restatement of Lemma 26: Let G'^S_ψ, G''^S_ψ be classically unsatisfiable MSCCs of G^S_ψ. If $G'^S_\psi \prec_S G''^S_\psi$, then $G'^S_\psi = G''^S_\psi$.

Proof: Since G'^S_ψ, G''^S_ψ are classically unsatisfiable, there exist some $\alpha, \neg\alpha \in V'^S_\psi$ and $\beta, \neg\beta \in V''^S_\psi$. Because $G'^S_\psi \prec_S G''^S_\psi$, we have $\alpha \prec_S \beta$ and $\neg\alpha \prec_S \neg\beta$. By Lemma 25 we obtain $\beta \prec_S \alpha$. Since $\alpha \prec_S \beta$ and $\beta \prec_S \alpha$, we finally get $G'^S_\psi = G''^S_\psi$. ■

C. Proof of Theorem 33

Here, we show that provided $J_l^1 = \emptyset$ and $J_l^2 \neq \emptyset$, 2-SAT in logics over De Morgan algebras is reducible to 2-SAT in Belnap's logic. First we prove some necessary lemmas and then we turn to proof of Theorem 33.

Lemma 37: Let $\|\psi\|_\rho \sqsupseteq l$, and $i \in J_2^l$. Define $\rho_\mathcal{B}$ in such a way that $Both \in \mathcal{I}_\mathcal{B}(\rho_\mathcal{B}(x))$ iff $i \in \mathcal{I}(\rho(x))$, and $None \in \mathcal{I}_\mathcal{B}(\rho_\mathcal{B}(x))$ iff $\mathbf{neg}(i) \in \mathcal{I}(\rho(x))$. Then $\|\psi\|_{\rho_\mathcal{B}} \sqsupseteq T$.

Proof: First, we show that $i \in \mathcal{I}(\|(\neg x)\|_\rho)$ iff $Both \in \mathcal{I}_\mathcal{B}(\|(\neg x)\|_{\rho_\mathcal{B}})$, and $\mathbf{neg}(i) \in \mathcal{I}(\|(\neg x)\|_\rho)$ iff $None \in \mathcal{I}_\mathcal{B}(\|(\neg x)\|_{\rho_\mathcal{B}})$.

We prove the first part, a proof of the second one is symmetric. If $i \in \mathcal{I}(\|(\neg x)\|_\rho)$, then $i \in \mathcal{I}(\neg \|x\|_\rho)$. By Definition 13 and Theorem 15, we have that $\mathbf{neg}(i) \notin \mathcal{I}(\|x\|_\rho)$. From definition of $\rho_\mathcal{B}$ follows that $None \notin \mathcal{I}_\mathcal{B}(\|x\|_\rho)$, and hence $Both \in \mathcal{I}_\mathcal{B}(\|(\neg x)\|_\rho)$. The reverse implication may be proved using similar arguments.

Suppose that, on the contrary, $\|\psi\|_\rho \sqsupseteq l$ and ρ_B does not satisfy ψ for B. Then there exists a clause $(\alpha \vee \beta)$ that does not satisfy: $\{Both, None\} \subseteq \mathcal{I}_\mathcal{B}(\|(\alpha \vee \beta)\|_{\rho_\mathcal{B}})$. Further, by definition of ρ_B, it does not hold that $\{i, \mathbf{neg}(i)\} \subseteq \mathcal{I}(\|(\alpha \vee \beta)\|_\rho)$. Therefore ρ does not satisfy ψ for l. Thus we obtained the contradiction and proved the lemma. ■

Lemma 38: Let $\|\psi\|_\rho \sqsupseteq l$, and $i \in J_2^l$. Then for every clause $(\alpha \vee \beta)$ such that α and β belong to some classically unsatisfiable component, we have that: $i \in \mathcal{I}(\|\alpha\|_\rho) \setminus \mathcal{I}(\|\beta\|_\rho)$, and $\mathbf{neg}(i) \in \mathcal{I}(\|\beta\|_\rho) \setminus \mathcal{I}(\|\alpha\|_\rho)$ or $i \in \mathcal{I}(\|\beta\|_\rho) \setminus \mathcal{I}(\|\alpha\|_\rho)$, and $\mathbf{neg}(i) \in \mathcal{I}(\|\alpha\|_\rho) \setminus \mathcal{I}(\|\beta\|_\rho)$.

Proof: Assume on the contrary that $\|\psi\|_\rho \sqsupseteq l$ and there is a clause $(\alpha \vee \beta)$, for which the above condition does not hold. Valuation ρ satisfies ψ for l so $\|\alpha \vee \beta\|_\rho \sqsupseteq l$ and in consequence $\{i, \mathbf{neg}(i)\} \subseteq \mathcal{I}(\|\alpha\|_\rho)$ or $\{i, \mathbf{neg}(i)\} \subseteq \mathcal{I}(\|\beta\|_\rho)$.

Consider the first case, the second one is analogical. Define $\rho_\mathcal{B}$ in the same way as in Lemma 37. Then $\|\psi\|_{\rho_\mathcal{B}} = T$, $\mathcal{I}_\mathcal{B}(\|\alpha\|_{\rho_\mathcal{B}}) \supseteq \{Both, None\}$ and in consequence $\|\alpha\|_{\rho_\mathcal{B}} = T$. But since α, β belong to some classically unsatisfiable component, it contradicts Lemma 22. ■

Lemma 39: Let $\|\psi\|_\rho \sqsupseteq l$, and $i \in J_3^l$. Let $\rho_\mathcal{B}$ satisfy: $Both \in \mathcal{I}_\mathcal{B}(\rho_\mathcal{B}(x))$ iff $i \in \mathcal{I}(\rho(x))$, and $None \in \mathcal{I}_\mathcal{B}(\rho_\mathcal{B}(x))$ iff $\mathbf{neg}(i) \in \mathcal{I}(\rho(x))$. Then $\|\psi\|_\rho \sqsupseteq Both$.

Proof: The proof may be done analogically to that for Lemma 37. ■

Lemma 40: Let ψ be a classically unsatisfiable 2-CNF formula. Let $\|\psi\|_\rho \sqsupseteq l$, and $i \in J_3^l$. Then for at least one literal α of each clause $\alpha \vee \beta$, where α, β belong to some classically unsatisfiable component of G_ψ^S, we have: $i \in \mathcal{I}(\|\alpha\|_\rho)$, and $\mathbf{neg}(i) \notin \mathcal{I}(\|\alpha\|_\rho)$.

Proof: The proof is similar to that of Lemma 38. It is enough to assume the contrary and to show that it contradicts Lemma 30. ■

Restatement of Theorem 33: Let $J_1^l = \emptyset$, and $J_2^l \neq \emptyset$. Let ψ be a classically unsatisfiable 2-CNF formula. Then, there exists ρ satisfying ψ for l iff all of the following holds.

1) There exist elements $\mathbf{l_1}, \mathbf{l_2} \in L$ such that:
   a) for every $i \in J_2^l$, we have $i \in \mathcal{I}(\mathbf{l_1}) \setminus \mathcal{I}(\mathbf{l_2})$, and $\mathbf{neg}(i) \in \mathcal{I}(\mathbf{l_2}) \setminus \mathcal{I}(\mathbf{l_1})$ or $\mathbf{neg}(i) \in \mathcal{I}(\mathbf{l_1}) \setminus \mathcal{I}(\mathbf{l_2})$, and $i \in \mathcal{I}(\mathbf{l_2}) \setminus \mathcal{I}(\mathbf{l_1})$;
   b) for every $i \in J_3^l$, we have $i \in \mathcal{I}(\mathbf{l_1})$, and $\mathbf{neg}(i) \notin \mathcal{I}(\mathbf{l_1})$ or $i \in \mathcal{I}(\mathbf{l_2})$, and $\mathbf{neg}(i) \notin \mathcal{I}(\mathbf{l_2})$.
2) There exists valuation $\rho_\mathcal{B}$ satisfying ψ for T.

Proof: Assume that $\|\psi\|_\rho \sqsupseteq l$. Let $\alpha \vee \beta$ be a clause of ψ such that α, β belong to some classically unsatisfiable component of G_ψ^S. Then by lemmas 38 and 40, we can choose $\|\alpha\|_\rho$ and $\|\beta\|_\rho$ for $\mathbf{l_1}$ and $\mathbf{l_2}$, resp. Moreover, from Lemma 37 follows $\|\psi\|_{\rho_\mathcal{B}} \sqsupseteq T$.

Now assume that there exist $\mathbf{l_1}, \mathbf{l_2} \in \mathcal{L}$ as defined above and that $\rho_\mathcal{B}$ satisfies ψ for T. Define $\rho(x) = \mathbf{l_1}$ iff $\rho_\mathcal{B}(x) = Both$, and $\rho(x) = \mathbf{l_2}$ iff $\rho_\mathcal{B}(x) = None$. Moreover, define $\rho(x) = \top$ iff $\rho_\mathcal{B}(x) = T$, and $\rho(x) = \bot$ iff $\rho_\mathcal{B}(x) = F$.

Claim 41: Using Theorem 15 and Definition 13, it may be shown that pairs $\neg\mathbf{l_1}, \neg\mathbf{l_2}$ and $\neg\mathbf{l_1}, \mathbf{l_2}$, and $\mathbf{l_1}, \neg\mathbf{l_2}$ also satisfy conditions 1a and 1b. Moreover, for all the pairs l', l'' that satisfy these conditions we have $\mathcal{I}(l') \cup \mathcal{I}(l'') \supseteq \mathcal{I}(l)$, and in consequence $l' \sqcup l'' \sqsupseteq l$.

Now, similarly to the proof of Lemma 27, we can show that $\|\psi\|_\rho \sqsupseteq l$. The one very different case is for a clause $(\alpha \vee \beta)$, where α, β belong to some classically unsatisfiable component of G_ψ^S. Therefore we consider this case and omit the others. Because of the claim above, we can restrict ourselves to the case $(x \vee y)$ i.e., where α is some variable x and β is some variable y. By Lemma 27, we know that $\|x\|_{\rho_\mathcal{B}} = Both$ and $\|y\|_{\rho_\mathcal{B}} = None$, and in consequence $\rho(x) = \mathbf{l_1}$ and $\rho(y) = \mathbf{l_2}$. (The second case is symmetric.) From the claim above we have $\mathbf{l_1} \sqcup \mathbf{l_2} \sqsupseteq l$ so $\|x \vee y\|_\rho \sqsupseteq l$. ■

D. Proofs of Theorems 35 and 36

Before we turn to proofs of theorems, we show that the function $\mathbf{neg}$ defined in Definition 13 is equivalent to a function with the same name from [9] and therefore may be defined as a value of a simple expression.

Lemma 42: Let $j \in \mathcal{J}(\mathcal{L})$, and let $\mathbf{neg}$ be function from Definition 13. Then $\mathbf{neg}(j) = \sqcap(L \setminus \downarrow \neg j)$, and $\mathbf{neg}(j)$ may be computed in time $O(|L|)$.

Proof: To see that $\sqcap(L \setminus \downarrow \neg j)$ may be computed in linear time observe that:

- valuation of $(\downarrow \neg j)$ returns set of size $O(|L|)$;
- operations $\bigsqcap A$ and $A_1 \setminus A_2$ for $|A|, |A_1|, |A_2| \leq |L|$ return sets of size $O(|L|)$ and may be computed in linear time.

Now, we show that $\mathbf{neg}(j) = \sqcap(L \setminus \downarrow \neg j)$. By Definition 13 and Theorem 15, it is enough to show that $\{\mathbf{neg}(j)\} = \bigcap(\mathcal{J}(\mathcal{L}) \setminus \downarrow (\mathcal{J}(\mathcal{L}) \setminus \{\mathbf{neg}(j)\}))$. It holds since $\downarrow (\mathcal{J}(\mathcal{L}) \setminus \{\mathbf{neg}(j)\}) = \{S \subseteq \mathcal{J}(\mathcal{L}) | \mathbf{neg}(j) \notin S\}$ and $\bigcap\{S \subseteq \mathcal{J}(\mathcal{L}) | \mathbf{neg}(j) \in S\} = \{\mathbf{neg}(j)\}$ ■

Restatement of Theorem 35: Algorithm 34 is sound and complete.

Proof: First, we prove soundness, that is, we show that if Algorithm 34 returns ρ, then $\|\psi\|_\rho \sqsupseteq l$. In the case of ρ constructed in point 8, it follows from Lemma 32. Correctness of the valuation constructed in points 11 and 13 follows from proofs of Lemma 27 and Theorem 33.

Now, we turn to completeness, that is, we show that if Algorithm 34 returns *no*, then there is no satisfying ρ. There are three points to consider: 7, 9 and 10. The case of point 7 follows from Lemma 31. In the remaining two cases either item 1 or item 2 of Theorem 33 is not satisfied. ■

Restatement of Theorem 36: Algorithm 34 works in time $O(|\psi| + |L|^2)$.

Proof: First we consider items whose complexity depends on $|\Gamma|$. Algorithm 34 uses intensively some well-known linear procedures: Tarjan algorithm [12] at point 1, algorithm for classical 2-SAT [2] at points 2, 13 and algorithm for 2-colouring at points 10, 11. Therefore all those items may be done in time $O(|\psi|)$.

Now we turn to items whose complexity depends only on $|L|$. To handle those entries efficiently, construct first a graph $G_\mathcal{L} = (L, \sqsubseteq)$. It takes time $O(|L|^2)$. By breadth-first search on $G_\mathcal{L}$ we can easily compute $\mathcal{J}(\mathcal{L})$ (entry 3). Join irreducibles are those vertices of $G_\mathcal{L}$ whose fan-in is exactly one. To obtain $I(l')$ for some $l' \in L$, it is enough to check, for every $j \in \mathcal{J}(\mathcal{L})$ whether $j \sqsubseteq l'$. Therefore point 4 may be done in time $O(|L|^2)$. By lemma 42, point 5 may be obtained in the same time. Point 6 may be clearly computed linearly in $|L|$.

■