Tractable Quantified Constraint Satisfaction Problems over Positive Temporal Templates*

Witold Charatonik and Michał Wrona

Institute of Computer Science
University of Wrocław

Abstract. A positive temporal template (or a positive temporal constraint language) is a relational structure whose relations can be defined over a dense linear order of rational numbers using a relational symbol $\leq$, logical conjunction and disjunction.

We provide a complexity characterization for quantified constraint satisfaction problems (*QCSP*) over positive temporal languages. The considered *QCSP* problems are decidable in LOGSPACE or complete for one of the following classes: NLOGSPACE, P, NP, PSPACE. Our classification is based on so-called algebraic approach to constraint satisfaction problems: we first classify positive temporal languages depending on their surjective polymorphisms and then give the complexity of *QCSP* for each obtained class.

The complete characterization is quite complex and does not fit into one paper. Here we prove that *QCSP* for positive temporal languages is either NP-hard or belongs to P and we give the whole description of the latter case, that is, we show for which positive temporal languages the problem *QCSP* is in LOGSPACE, and for which it is NLOGSPACE-complete or P-complete. The classification of NP-hard cases is given in a separate paper.

1 Introduction

Constraint Satisfaction Problems provide a uniform approach to research on a wide variety of combinatorial problems. Besides probably better-known *CSP* over finite domains [9, 11, 19] with its Dichotomy conjecture of Feder and Vardi [12], *CSP* over infinite domains are of more and more interest. Although there were some earlier results in this field [1, 16], a common approach to *CSP* over infinite domains was quite recently proposed and developed by Manuel Bodirsky [2] and co-authors. This framework concentrates on relational structures that are ω-categorical [14]. Many results, including so-called algebraic approach [15, 11], for both *CSP* and *QCSP* [8] over finite domains were generalized to infinite ones. Moreover, new results were established. Among them there are full characterizations of complexity for both *CSP* and *QCSP* of equality constraint languages [6, 4].

As each natural theoretical framework *CSP* have many different applications. It is also the case in the area of *CSP* with infinite templates. For example, in [3, 5] it is argued that *CSP* of relations definable over the dense linear order of rational numbers

* Work partially supported by Polish Ministry of Science and Education grant 3 T11C 042 30

may constitute a good approach to temporal and spatial reasoning [13, 18]. Therefore the very recent full complexity characterization of *CSP* for temporal, that is, definable over $\langle \mathbb{Q}, < \rangle$ templates gives a new perspective on temporal reasoning [7]. One should mention in this context also the well-motivated AND/OR precedence constraints [17]. They are closely related to languages from items 3 and 4 of Theorem 1 below. It might be said that we consider quantified positive variations of AND/OR precedence constraints.

Our paper is the next step in the area of quantified constraint satisfaction problems over temporal templates. In general, we consider *QCSP* for temporal templates. In particular, we restrict ourselves to constraint languages that can be defined with $\wedge, \vee$ and $\leq$, i.e., we do not consider negation. We name such relations positive temporal since they are positive definable over $\langle \mathbb{Q}, \leq \rangle$.

Our main contribution is a complexity characterization of QCSP problems over positive temporal languages summarized in Theorem 1 below.

Theorem 1 (The Main Theorem). *Let Γ be a language of positive temporal relations, then one of the following holds.*

1. *Each relation in Γ is definable by a conjunction of equations ($x_1 = x_2$) and then QCSP(Γ) is decidable in LOGSPACE.*
2. *Each relation in Γ is definable by a conjunction of weak inequalities ($x_1 \leq x_2$). If there exists a relation in Γ that is not definable as a conjunction of equalities, then QCSP(Γ) is NLOGSPACE-complete.*
3. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ and then provided Γ satisfies neither condition 1 nor 2, the set QCSP(Γ) is P-complete.*
4. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$ and then provided Γ satisfies neither condition 1 nor 2, the set QCSP(Γ) is P-complete.*
5. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} = y_{i_1} \vee \ldots \vee x_{i_k} = y_{i_k})$ and then provided Γ does not belong to any of the classes 1–4, the set QCSP(Γ) is NP-complete.*
6. *The problem QCSP(Γ) is PSPACE-complete.*

The complete characterization is quite complex and does not fit into one paper. Therefore some parts of Theorem 1 are proved in a companion paper [10]. Technically, in this paper we show the following.

Theorem 2. *If Γ is a positive temporal language, then either it satisfies one of conditions 1–4 of Theorem 1 or QCSP(Γ) is NP-hard.*

Related work. As we already mentioned, parts of Theorem 1 are proved in a different paper [10]. We give there the characterization of NP-hard cases, with the distinction between NP-complete and PSPACE-complete cases (items 5 and 6 of Theorem 1) and the complexity proofs for items 3 and 4 of Theorem 1 (the algebraic characterization of these cases is given here in Section 5). In [10] we use different techniques, in particular we do not use there the filter representation of temporal relations which is our basic tool in this paper.

The characterizations of cases 1, 2 and 5 are due to other authors [4, 3]. Here, we just complete the picture by giving the NLOGSPACE-hardness proof for case 2. Theorem 1 substantially improves results from [4, 3] in the sense that we consider a strictly more expressive class of constraint languages. As in [4] we use the surjective preservation theorem.

In a very recent paper [7] the authors give a classification of *CSP* over temporal languages depending on their polymorphisms. Although it sounds similar, it is different from our classification. We deal with positive temporal languages and surjective polymorphisms, which are used to classify *QCSP* problems (as opposed to *CSP* problems considered in [7]). In the case of positive temporal languages, the classification based on polymorphisms is trivial: all these languages fall into the same class because they are all closed under constant functions — as a consequence all *CSP* problems for positive temporal languages are trivial. To obtain our classification we use methods different from those used in [7]. The representation of temporal relations from Section 3 and the proof in sections 5 and 6 plays a similar role in the proof of Theorem 1 as the Ramsey Theorem in the proof from [7] in the sense that it is used to show that there are only few interesting classes of positive temporal relations closed under non-unary functions. Nevertheless our representation may be easily translated into first order formula. It is very useful in the context of considered *QCSP* problems — complexity proofs for families of positive temporal templates from Theorem 1 are purely syntactical [4, 3, 10].

Outline of the paper. In Section 2, we give some preliminaries. Among others, we recall a definition of a surjective polymorphism and surjective preservation theorem, which is the most important tool in algebraic approach to *QCSP*. In Section 3 we propose a representation of temporal relations. Section 4 is devoted to formally introducing the concept of positive temporal relations. By giving their properties we unfold the reason we choose just that subset. In Section 5, we derive first four items of Theorem 1, in Section 6 we prove that all other languages are NP-hard and finally the last section is devoted to the proof of Theorem 2.

2 Preliminaries

In most cases we follow the notation from [2, 4].

Relational structures. We consider only relations defined over countable domains and hence whenever we write a domain or D we mean a countable set. Let τ be some relational (in this paper always finite) signature i.e., a set of relational symbols with assigned arity. Then Γ is a τ-structure over domain D if for each relational symbol R_i from τ, it contains a relation of according arity defined on D. In the rest of the paper we usually say relational language (or template) instead of relational structure. Moreover, we use the same notation for relational symbols and relations.

Automorphisms of Γ constitute a group with respect to composition. An orbit of a k-tuple t in Γ is the set of all tuples of the form $\langle \Pi(t_1), \ldots, \Pi(t_k) \rangle$ for all automorphisms Π. We say that a group of automorphisms of Γ is oligomorphic if for each k it has a finite number of orbits of k-tuples. Although there are many different ways of introducing a concept of ω-categorical structures we do it by the following theorem [14].

Theorem 3. ***(Engeler, Ryll-Nardzewski, Svenonius)*** *Let Γ be a relational structure. Then Γ is ω-categorical if and only if the automorphism group of Γ is oligomorphic.*

Polymorphisms and clones. Let R be a relation of arity n defined over D. We say that a function $f : D^m \to D$ is a polymorphism of R if for all $a^1, \ldots, a^m \in R$ (where a^i, for $1 \leq i \leq m$, is a tuple $\langle a^i_1, \ldots, a^i_n \rangle$), we have $\langle f(a^1_1, \ldots, a^m_1), \ldots, f(a^1_n, \ldots, a^m_n) \rangle \in R$. Then we say that f preserves R or that R is closed under f. A polymorphism of Γ is a function that preserves all relations of Γ. By *Pol*(Γ) we denote the set of polymorphisms of Γ, and by *sPol*(Γ) — the set of surjective polymorphisms.

An operation π is a projection iff $\pi(x_1, \ldots, x_m) = x_i$ for all m-tuples and fixed $i \in \{1, \ldots, m\}$. The set of polymorphisms of an ω-categorical language Γ constitute a *clone*, that is, a set of functions closed under composition and containing all projections. We say that a function f with a domain D is an *interpolation* of a set of functions F iff for every finite subset B of D there is some operation $g \in F$ such that $f(a) = g(a)$ for every tuple a over the set B. The set of interpolations of F is called the *local closure* of F. We say that a clone is *locally closed* if each its subset contains its local closure. For each ω-categorical language Γ the clone of its polymorphisms is locally closed [2]. A clone is generated by a set of functions F if it is the least clone containing F.

An operation f of arity m is *essentially unary* if there exists a unary operation f_0 such that $f(x_1 \ldots, x_m) = f_0(x_i)$ for some fixed $i \in \{1, \ldots, m\}$. An operation that is not essentially unary is called *essential*. We say that a polymorphism f of an ω-categorical structure Γ is *oligopotent* if the diagonal of f, that is, the function $f(x, \ldots, x)$, is contained in the locally closed clone generated by the automorphisms of Γ. A function f is called a *quasi near-unanimity function* (QNUF) if it satisfies $f(x, \ldots, x, y) = f(x, \ldots, y, x) = \ldots = f(y, x, \ldots, x) = f(x, \ldots, x)$ for all $x, y \in D$.

Quantified constraint satisfaction problems. Let Γ contain $R_1, \ldots, R_k$. Then a conjunctive positive formula (*cp-formula*) over Γ is a formula of the following form:

$$Q_1 x_1 \ldots Q_n x_n (R_1(\boldsymbol{v_1}) \wedge \ldots \wedge R_k(\boldsymbol{v_k})), \tag{1}$$

where $Q_i \in \{\forall, \exists\}$ and $\boldsymbol{v_j}$ are vectors of variables $x_1, \ldots, x_n$.

A *QCSP*(Γ) is a problem to decide whether a given cp-formula without free variables over Γ is true or not. Note that by downward Löwenheim-Skolem Theorem we can focus on countable domains only. If all quantifiers in (1) are existential then the corresponding problem is well-known as the *constraint satisfaction problem*.

A relation R has a *cp-definition* in Γ if there exists a cp-formula $\phi(x_1, \ldots, x_n)$ over Γ such that for all $a_1, \ldots, a_n$ we have $R(a_1, \ldots, a_n)$ iff $\phi(a_1, \ldots, a_n)$ is true. The set of all relations cp-definable in Γ is denoted by $[\Gamma]$.

Lemma 1 ([4]). *Let Γ_1, Γ_2 be relational languages. If every relation in Γ_1 has a cp-definition in Γ_2, then QCSP(Γ_1) is log-space reducible to QCSP(Γ_2).*

The following results link $[\Gamma]$ with *sPol*(Γ). The idea behind Theorem 4 is that the more Γ can express, in the sense of cp-definability, the less polymorphisms are contained in *sPol*(Γ). Moreover, the converse is also true. This theorem is called *surjective preservation theorem*.

Theorem 4 ([4]). *Let Γ be an ω-categorical structure. Then a relation R has a cp-definition in Γ if and only if R is preserved by all surjective polymorphisms of Γ.*

As a direct consequence of Lemma 1 and Theorem 4 we obtain the following.

Corollary 1 ([4]). *Let Γ_1, Γ_2 be ω-categorical structures. If sPol$(\Gamma_2) \subseteq$ sPol(Γ_1), then QCSP(Γ_1) is log-space reducible to QCSP(Γ_2).*

Quantified Equality Constraints. Concerning templates that allow equalities and all logical connectives (equality constraint languages) the following classification [4] is known.

1. **Negative languages**. Relations of such a language are definable as CNF-formulas whose clauses are either equalities ($x = y$) or disjunctions of disequalities ($x_1 \neq y_1 \vee \ldots \vee x_k \neq y_k$). For each negative Γ the problem *QCSP*(Γ) is contained in LOGSPACE.
2. **Positive languages**. Relations may be defined as conjunction of disjunctions of equalities ($x_1 = y_1 \vee \ldots \vee x_k = y_k$). For each positive Γ not being negative the problem *QCSP*(Γ) is NP-complete.
3. In any other case the problem *QCSP*(Γ) is PSPACE-complete.

Note that the class 1 from Theorem 1 is a subset of negative languages and the class 5 is just the class of positive languages.

To give our characterization we need the following result. It may be inferred from lemmas given in Section 7 in [4].

Lemma 2. *Let Γ be an equality positive constraint language that is preserved by an essential operation on D with infinite image. Then Γ is preserved by all operations, and Γ is negative.*

Corollary 2. *If an equality constraint language Γ is positive, but not negative, then sPol(Γ) contains only essentially unary polymorphisms.*

Since *QCSP*(Γ) for positive non-negative equality language Γ is NP-hard, by corollaries 1 and 2, we obtain one more observation.

Corollary 3. *Let Γ be a positive temporal language with sPol(Γ) contained in the set of essentially unary surjections on $\mathbb{Q}$. Then QCSP(Γ) is NP-hard.*

Quantified Positive Temporal Constraints. Now, we focus on positive temporal relations announced in the introduction. All of them are defined over the set of rational numbers using a relational symbol $\leq$ and connectives $\wedge, \vee$. Therefore our results concerning positive temporal relations generalize those for positive equality languages. Since the only relational symbol we use is interpreted as a weak linear order over rational numbers, for each positive temporal structure Γ the set *sPol*(Γ) contains all automorphisms that preserve order, i.e., all increasing unary surjections $f : \mathbb{Q} \rightarrow \mathbb{Q}$. We say that f is increasing (weakly increasing) if $f(a) > f(b)$ ($f(a) \geq f(b)$) for all $a > b$. Thus, using Theorem 3, it is not hard to see that all positive temporal languages are ω-categorical.

In Lemma 6 in Section 4 we provide another, not syntactical, characterization of positive temporal relations. It is given in terms of polymorphisms.

3 Filter Representation of Temporal Relations

Here, after a few definitions we give a representation of temporal languages that we use in the rest of the paper. At first look it may look somewhat confusing, but Example 1 and a short discussion after it should clarify our point.

A preorder is a reflexive and transitive relation. A preorder $\preceq$ on a set A is total if for all $a, b \in A$ we have $a \preceq b$ or $b \preceq a$. We call A the domain of $\preceq$ and we write $A = Dom(\preceq)$. We use $a \prec b$ as an abbreviation for $a \preceq b \wedge b \npreceq a$ and $a \approx b$ as an abbreviation for $a \preceq b \wedge b \preceq a$. In the following we represent total preorders on finite sets of variables as sequences of the form $x_1 \sim_1 x_2 \sim_2 \ldots \sim_{n-1} x_n$ where each $\sim_i$ is either $\prec$ or $\approx$ and $\{x_1, \ldots, x_n\} = Dom(\preceq)$. For example $a \prec b \approx c$ is the representation of $\preceq = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle, \langle c, b\rangle\}$. By *Range*$(\preceq)$ we denote the size of a maximal set of variables $\{x_{i_1}, \ldots, x_{i_l}\} \subseteq Dom(\preceq)$ such that for all pairs x_{i_p}, x_{i_r} we have either $x_{i_p} \prec x_{i_r}$ or $x_{i_r} \prec x_{i_p}$.

We write $\preceq_1 \lhd \preceq_2$ and say that $\preceq_1$ is more general than $\preceq_2$ if $\preceq_1$ is a restriction of $\preceq_2$ to a smaller domain. Formally, $\preceq_1 \lhd \preceq_2$ if $Dom(\preceq_1) \subseteq Dom(\preceq_2)$ and $\preceq_1 = \preceq_2 \cap (Dom(\preceq_1) \times Dom(\preceq_1))$. We write $\preceq_1 \ll \preceq_2$ and say that $\preceq_1$ is flatter than $\preceq_2$ if $Dom(\preceq_1) = Dom(\preceq_2)$ and $\preceq_2$ as a relation is a subset of $\preceq_1$ (see the example below).

We say that a valuation $q : \{x_1, \ldots, x_n\} \rightarrow \mathbb{Q}$ is compatible with a total preorder $\preceq$ on $\{x_1, \ldots, x_n\}$ if for all x_i, x_j such that $x_i \preceq x_j$ we have $q(x_i) \leq q(x_j)$. We then also say that the tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ is compatible with $\preceq$.

Definition 1. *Consider a temporal relation $R(x_1, \ldots, x_n) \subseteq \mathbb{Q}^n$. We say that $\preceq$ is a* bound *for R if $\preceq$ is a total preorder on a subset of $\{x_1, \ldots, x_n\}$ such that for all valuations $q : \{x_1, \ldots, x_n\} \rightarrow \mathbb{Q}$ compatible with $\preceq$ the tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ is* not *in R. The set of bounds of R is denoted $\mathcal{B}(R)$. A minimal wrt. $\lhd$ bound of R is called a* filter *for R. The set of filters of R is denoted $\mathcal{F}(R)$.*

Let Γ be a temporal template. Assume that each $R \in \Gamma$ is defined over different set of variables. Then by $\mathcal{F}(\Gamma)$ equal to $\bigcup_{R \in \Gamma} \mathcal{F}(R)$ we denote the set of filters of Γ. Similarly, we write $\mathcal{B}(\Gamma)$ for the set of bounds of Γ.

Example 1. Let $R(x_1, x_2, x_3)$ be a relation given by $(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_2 \leq x_1)$. Then $x_3 \prec x_2 \prec x_1$, $x_1 \prec x_2$ and $x_1 \approx x_3 \prec x_2$ are bounds for R (in fact R has more bounds). The bound $x_1 \approx x_3 \prec x_2$ is not a filter because $x_1 \prec x_2$ is more general. The relation R' defined by $(x_1 \leq x_2 \vee x_1 \leq x_3)$ has three filters: $x_3 \prec x_2 \prec x_1$, $x_3 \approx x_2 \prec x_1$ and $x_2 \prec x_3 \prec x_1$. The filter $x_3 \approx x_2 \prec x_1$ is flatter than both $x_3 \prec x_2 \prec x_1$ and $x_2 \prec x_3 \prec x_1$. The range of a preorder $x_3 \approx x_2 \prec x_1$ is 2, but the range of a preorder $x_3 \prec x_2 \prec x_1$ is equal to 3.

In the following we represent temporal relations (languages) with their sets of filters. It is quite simple to infer $\mathcal{F}(R)$ from the cp-definitions in Example 1. Therefore one would ask why we do not represent relations with their sets of clauses. The filter representation gives us a kind of normal form while there may be many representations of the same relations with sets of clauses. The following example shows another advantage of filters. The relation R defined by $(x_1 \geq x_2 \vee x_2 \geq x_3)$ has one filter: $x_1 \prec x_2 \prec x_3$. The clause representation $(x_1 \geq x_2 \vee x_2 \geq x_3)$ does not say anything about dependencies between x_1 and x_3. From the shape of the filter it is easy to see that R contains tuples compatible with preorders: $x_1 \prec x_2, x_1 \prec x_3, x_2 \prec x_3$ but it does not contain any tuple compatible with $x_1 \prec x_2 \prec x_3$. We use this property in several proofs.

Consider now the situation where we have some $\mathcal{F}(\Gamma)$ and we ask for $\mathcal{F}(R)$ of some relation R that is cp-definable over Γ; and the converse situation: when we want to infer something about $\mathcal{F}(\Gamma)$ from $\mathcal{F}(R)$. The following lemmas give us a partial answer for such questions. In lemmas 3–5 the relation R_1 belongs to $[R]$. These lemmas are used in each of the following sections of the paper and therefore are of crucial importance.

Let $\preceq_{\mathit{Var}}$ be a preorder with domain *Var* such that $x \approx y$ for all $x, y \in \mathit{Var}$.

Lemma 3. *Let $R(x_1, \ldots, x_n)$ be a temporal relation. Consider $R_1(x_k, \ldots, x_n)$ defined by $R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{k-1} x_i = x_{i+1}$. If a preorder $\preceq_1$ is a bound of R_1, then each preorder $\preceq$ such that $\preceq_1 \lhd \preceq$ and $x_i \approx x_{i+1}$ for $1 \leq i \leq k-1$ is a bound of R.*

Lemma 4. *Let $R(x_1, \ldots, x_n)$ be a temporal relation with a filter $\preceq$ such that $x_i \approx x_{i+1}$ for $i = 1, \ldots, k-1$. Consider $R_1(x_k, \ldots, x_n)$ defined by $R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{k-1} x_i = x_{i+1}$. Then either the restriction of $\preceq$ to $\{x_k, \ldots, x_n\}$ or the restriction of $\preceq$ to $\{x_{k+1}, \ldots, x_n\}$ is a filter of R_1.*

Example 2. Consider the preceding lemma once more. At first glance it may seem that a restriction of $\preceq$ to $\{x_k, \ldots, x_n\}$ is always a filter of R_1 and the second case is unnecessary there. To see that it is not the case, consider the following relation.

Let $R(x_1, x_2, x_3, x_4)$ be defined by $(x_1 = x_2 \vee x_1 = x_3 \vee x_1 = x_4) \wedge (x_2 = x_1 \vee x_2 = x_3 \vee x_2 = x_4) \wedge (x_3 = x_1 \vee x_3 = x_2 \vee x_3 = x_4) \wedge (x_4 = x_1 \vee x_4 = x_2 \vee x_4 = x_3)$. Then $x_1 \approx x_2 \approx x_3 \prec x_4$ is a filter of R. Moreover, let a relation $R_1(x_2, x_3, x_4)$ be equivalent to $R(x_1, x_2, x_3, x_4) \wedge x_1 = x_2$. Now, the reader can convince himself that $x_3 \prec x_4$ and not $x_2 \approx x_3 \prec x_4$ is a filter of R_1.

Lemma 5. *Consider $R(x_1, \ldots, x_n)$ and $R_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ defined by $\exists x_i\ R(x_1, \ldots, x_n)$. Then a preorder $\preceq$ such that $x_i \notin Dom(\preceq)$ is a filter of R if and only if it is a filter of R_1.*

4 Positive Temporal Relations

An arbitrary relation defined over the dense linear order of rational numbers is closed under all unary strictly increasing functions — they are automorphisms of such a relation. In the case of positive temporal relations we have a bit stronger result.

Lemma 6. *A temporal relation $R(x_1, \ldots, x_n)$ is positive if and only if it is closed under all weakly increasing functions $f : \mathbb{Q} \to \mathbb{Q}$.*

It is rather obvious that positive temporal relations have substantially less varied structure than arbitrary temporal relations. The dependencies between tuples in positive temporal relations can be given in terms of the order $\ll$. The following results unfold the reason why we introduced this order. Recall: if $\preceq \ll \preceq_1$, then we say that $\preceq$ is flatter than $\preceq_1$.

Lemma 7. *Let a preorder $\preceq$ be flatter than $\preceq_1$ and $Dom(\preceq) \subseteq \{x_1, \ldots, x_n\}$. Consider a tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ compatible with $\preceq_1$ where $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$. Then there exists a weakly increasing function f such that $\langle f(q(x_1)), \ldots, f(q(x_n))\rangle$ is compatible with $\preceq$.*

Lemma 8. *Let $R(x_1, \ldots, x_n)$ be a temporal relation. Then it is a positive temporal relation if and only if for all bounds $\preceq$ of R each preorder $\preceq_1$ such that $\preceq \ll \preceq_1$ is also a bound of R.*

Corollary 4. *Let $R(x_1, \ldots, x_n)$ be a positive temporal relation and let the set Var be a nonempty subset of $\{x_1, \ldots, x_n\}$. Then the preorder $\preceq_{Var}$ is a bound of R if and only if R is the empty relation. Intuitively, nonempty relations do not have filters of the form $\preceq_{Var}$.*

Note that if we have two bounds comparable wrt $\ll$ then by Lemma 8 the flatter of them is more interesting. Therefore we focus on minimal wrt $\ll$ filters. We illustrate this lemma on the following example.

Example 3. Consider once more a positive temporal relation given by $(x_1 \leq x_2 \vee x_1 \leq x_3)$. It has a filter $x_2 \approx x_3 \prec x_1$, but also filters (bounds) $x_2 \prec x_3 \prec x_1$ and $x_3 \prec x_2 \prec x_1$. Note or recall from Example 1 that the first of the mentioned preorders is flatter than the remaining two. Moreover, these are all filters of the relation $(x_1 \leq x_2 \vee x_1 \leq x_3)$.

We finish this section by one more auxiliary lemma.

Lemma 9. *Let $R(x_1, \ldots, x_n)$ be a temporal relation. Let $\preceq$ be a minimal with respect to $\ll$ filter of R such that $x_i \approx x_{i+1}$ for $i = 1, \ldots, k-1$. Let $R_1(x_k, \ldots, x_n)$ be defined as $R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{k-1} x_i = x_{i+1}$. Then there exists a preorder $\preceq_1$ in $\mathcal{F}(R_1)$ such that $\preceq_1 \lhd \preceq$ and Range$(\preceq_1)$ = Range$(\preceq)$.*

5 Non Unary Surjective Polymorphisms of Positive Temporal Relations

In this section we derive the first four classes of Theorem 1. We show that a positive temporal language belongs to one of these four classes, or it is closed under essentially unary surjective polymorphisms only and hence, by Corollary 3, in that case the problem $\mathit{QCSP}(\Gamma)$ is NP-hard.

In particular we show that a positive temporal language is closed under a binary surjective polymorphism *spp*, under a binary surjective polymorphism *dual-spp*, or it is preserved by essentially unary surjective polymorphisms only. The surjective polymorphisms $\mathit{spp} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ and $\mathit{dual\text{-}spp} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ are surjective counterparts of *pp* and *dual-pp* operations introduced in [7].

Recall that all countable dense linear orders without endpoints are isomorphic. In particular, $\mathbb{Q}, \mathbb{Q}_-, \mathbb{Q}_+$ and $\mathbb{Q}_- \cup \mathbb{Q}_+$ are isomorphic, where $\mathbb{Q}_- = \{q \in \mathbb{Q} \mid q < 0\}$ and $\mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q > 0\}$. Let $f_1 : \mathbb{Q} \to \mathbb{Q}_-$, $f_2 : \mathbb{Q} \to \mathbb{Q}_+$ and $f : \mathbb{Q}_- \cup \mathbb{Q}_+ \to \mathbb{Q}$ be any order-preserving bijections. Let

$$\mathit{spp}'(a,b) = \begin{cases} a & \text{if } a < 0 \\ f_2(b) & \text{if } a \geq 0 \end{cases} \qquad \mathit{dual\text{-}spp}'(a,b) = \begin{cases} f_1(b) & \text{if } a \leq 0 \\ a & \text{if } a > 0 \end{cases}$$

and define $\mathit{spp}(a,b) = f(\mathit{spp}'(a,b))$ and $\mathit{dual\text{-}spp}(a,b) = f(\mathit{dual\text{-}spp}'(a,b))$. Observe that if $\mathit{spp}(a,b)$ is a (strict) lower bound of $\{\mathit{spp}(a_1,b_1),\ldots,\mathit{spp}(a_k,b_k)\}$ then either a is a (strict) lower bound of $\{a_1,\ldots,a_k\}$ or b is a (strict) lower bound of $\{b_1,\ldots,b_k\}$. Similarly, if $\mathit{dual\text{-}spp}(a,b)$ is a (strict) upper bound of the set $\{\mathit{dual\text{-}spp}(a_1,b_1),\ldots,$ $\mathit{dual\text{-}spp}(a_k,b_k)\}$ then either a is a (strict) upper bound of $\{a_1,\ldots,a_k\}$ or b is a (strict) upper bound of $\{b_1,\ldots,b_k\}$.

For each $i \geq 2$ let R^i_{Left} be the positive temporal relation defined by the formula $(x_1 \leq x_2 \vee \ldots \vee x_1 \leq x_i)$. Let Γ_{Left} be the positive temporal language containing R^i_{Left} for each i. Similarly, Γ_{Right} is the set of relations defined by formulas $(x_2 \leq x_1 \vee \ldots \vee x_i \leq x_1)$. Each such formula is denoted by R^i_{Right}.

5.1 Classes of different power of cp-definability

The topic of this subsection is summarized by the following theorem. Recall from Section 2 that $[\Gamma]$ is the set of all relations that are cp-definable by relations of Γ. Note that by Lemma 1, the problems $\mathit{QCSP}(\Gamma_1)$ and $\mathit{QCSP}(\Gamma_2)$ for some positive temporal languages Γ_1 and Γ_2 such that $[\Gamma_1] = [\Gamma_2]$ are logspace equivalent.

Theorem 5. *Let Γ be a positive temporal language, then exactly one of the following holds.*

1. *Each relation in Γ is definable by a conjunction of equations $(x_1 = x_2)$ and then $[\Gamma]$ is equal to $[x = y]$.*
2. *Each relation in Γ is definable by a conjunction of inequalities $(x_1 \leq x_2)$ and then $[\Gamma]$ is equal to $[x \leq y]$.*

3. *Each relation in Γ is definable by the formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ and then provided Γ satisfies neither condition 1 nor 2, the set $[\Gamma]$ is equal to $[\Gamma_{Left}]$.*
4. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$ and then provided Γ satisfies neither condition 1 nor 2, the set $[\Gamma]$ is equal to $[\Gamma_{Right}]$.*
5. *Each relation in Γ is preserved by essentially unary surjective polymorphisms only.*

First, we want to distinguish $[\Gamma_{Left}]$ and $[\Gamma_{Right}]$ from each other and then from $[(x_1 \leq x_2)]$. Recall that in our language, see Theorem 4, sets $[\Gamma_1]$ and $[\Gamma_2]$ for some positive temporal relations Γ_1 and Γ_2 are different if there exists a function that preserves relations from exactly one of these sets.

Lemma 10. *The language Γ_{Left} is closed under dual-spp operation but it is not closed under spp. Dually, Γ_{Right} is closed under spp but it is not closed under dual-spp.*

It is quite obvious that $(x_1 \leq x_2)$ is closed under both *spp* and *dual-spp*. In [3], it is shown that $(x_1 \leq x_2)$ is closed under *median*, which is the ternary function that returns the median of its three argument. It is not hard to show that *median* is a surjective oligopotent QNU polymorphisms. To distinguish $(x_1 \leq x_2)$ from Γ_{Left} and Γ_{Right} we show that the last two relations are not closed under any surjective QNU polymorphism.

Lemma 11. *Neither Γ_{Left} nor Γ_{Right} is closed under any surjective oligopotent QNU polymorphism.*

It turns out that the relation defined by $(x_1 \leq x_2 \vee x_1 \leq x_3)$ has the same expressive power, in the sense of cp-definability, as Γ_{Left}. In the same context the relation defined by $(x_2 \leq x_1 \vee x_3 \leq x_1)$ is as powerful as the whole Γ_{Right}.

Lemma 12. *Every relation in Γ_{Left} has a cp-definition over $(x_1 \leq x_2 \vee x_1 \leq x_3)$, that is, $[(x_1 \leq x_2 \vee x_1 \leq x_3)] = [\Gamma_{Left}]$. Similarly, $[(x_2 \leq x_1 \vee x_3 \leq x_1)] = [\Gamma_{Right}]$.*

Consider the following forms of filters.

$$z_1 \approx \ldots \approx z_k \prec y_1 \tag{2}$$

$$z_1 \prec y_1 \approx \ldots \approx y_k \tag{3}$$

We say that a preorder $\preceq$ with domain $Dom(\preceq) = \{x_1, \ldots, x_n\}$ is of the form, let's say, (2) if $n = k + 1$ and $x_{i_1} \approx \ldots \approx x_{i_{n-1}} \prec x_{i_n}$ for some permutation of $\{x_1, \ldots, x_n\}$.

The next theorem shows us the difference between positive temporal languages with non-unary and only unary polymorphism. This difference is expressed using their filters.

Theorem 6. *Let Γ be a positive temporal language. Consider the following conditions.*

1. *All filters minimal with respect to $\ll$ in Γ are of the form (2).*
2. *All filters minimal with respect to $\ll$ in Γ are of the form (3).*

If neither of these conditions holds, then sPol(Γ) contains only essentially unary polymorphisms.

The proof of this theorem is presented in the next section. Here, we show that the expressive power of a positive temporal template satisfying item 1 of Theorem 6 is not higher than the one of $(x_1 \leq x_2 \vee x_1 \leq x_3)$. A similar statement concerning languages satisfying item 2 of Theorem 6 and $(x_2 \leq x_1 \vee x_3 \leq x_1)$ is also true.

Lemma 13. *Let Γ be a positive temporal language.*

1. *If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form $z_1 \prec y_1$, then $[\Gamma] \subseteq [x_1 \leq x_2]$.*
2. *If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form (2) and at least one of them has a domain of size greater than or equal to 3, then $[\Gamma] = [\Gamma_{Left}]$.*
3. *If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form (3) and at least one of them has a domain of size greater than or equal to 3, then $[\Gamma] = [\Gamma_{Right}]$.*
4. *If $[\Gamma] \subseteq [x_1 \leq x_2]$, then either $[\Gamma] = [x_1 \leq x_2]$ or $[\Gamma] = [x_1 = x_2]$.*

Proof of Theorem 5. (Part One) First we show for a positive temporal language Γ that either $[\Gamma]$ is equal to exactly one of the following

1. $[x_1 \leq x_2 \vee x_1 \leq x_3]$
2. $[x_2 \leq x_1 \vee x_3 \leq x_1]$
3. $[x_1 \leq x_2]$
4. $[x_1 = x_2]$

or Γ is closed under essentially unary polymorphisms only. By Theorem 6, it is enough to prove that if all filters minimal with respect to $\ll$ of Γ are either of the form (2) or of the form (3), then $[\Gamma]$ is equal to exactly one of the above classes.

By Lemma 13 we obtain that $[\Gamma]$ is equal to at least one of them. To show that these classes are pairwise disjoint we use appropriate polymorphisms from the preceding lemmas and Theorem 4. By Lemma 10 we have that the first of the above families is closed under *dual-spp* operation but is not preserved by *spp*. By Lemma 11, it is not closed under any surjective oligopotent QNU polymorphism. Using the same lemmas we obtain a similar statement about the second family. Further, from [3] we know that the third family is closed under a surjective oligopotent QNU polymorphism. To distinguish the third and the fourth of the above sets using a function (in fact a permutation of rational numbers) note that the former is not closed under any strictly decreasing unary function.

(Part Two) Since all the classes are pairwise disjoint we can infer from Lemma 13 that $[\Gamma] = [\Gamma_{Left}]$ if and only if all filters from Γ are of the form (2) and at least one of them has a domain of size at least 3. Now, we show that if it is the case, then each relation from Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$. Indeed it is enough to introduce a clause $(y_1 \leq y_2 \vee \ldots \vee y_1 \leq y_m)$ for each filter of the form $y_2 \approx \ldots \approx y_m \prec y_1$ from $\mathcal{F}(R)$. Now it is not hard to prove that a valuation q does not satisfy such a clause if and only if q is compatible with some preorder more general than $y_2 \approx \ldots \approx y_m \prec y_1$. Because it is a filter of R, by Lemma 8 we are done.

Similarly we can prove that if $[\Gamma] = [\Gamma_{Right}]$, then each relation from Γ may be defined by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$.

Further, if $[\Gamma]$ is equal to $[x_1 \leq x_2]$, then all filters are of the form $z \prec y$. Here, we can show that each relation in Γ is definable by a conjunction of inequalities.

Finally, if $[\Gamma]$ is equal to $[x_1 = x_2]$, then, by Lemma 13, all filters are of the form $z \prec y$ and for each such a filter the set $\mathcal{F}(\Gamma)$ contains $y \prec z$ as well. Therefore it is not hard to show that each relation in Γ may be defined as a conjunction of equalities. $\square$

6 Proof of Theorem 6

Although the last section contains the proof of Theorem 5, Theorem 6 was left without an explanation. This section is devoted to fill this hole.

The idea behind the proof is to show that if a positive temporal template Γ contains a filter that is not of the form (2) and a filter that is not of the form (3), then $[\Gamma]$ contains some positive non-negative equality relation R. By Corollary 2, the relation R and hence, by Theorem 4, the language Γ is closed only under essentially unary surjections.

To prove this we consider a few cases. In most of them, we use the following lemma.

If an n-ary positive temporal relation $R(x_1, \ldots, x_n)$ is different from $\mathbb{Q}^n$ and contains $\bigvee_{i \neq j} x_i = x_j$ for $1 \leq i, j \leq n$ as a subrelation then we call it potentially non-negative positive.

Lemma 14. *Let R be a potentially non-negative positive relation. Then it is closed under essentially unary polymorphism only.*

Consider the followings forms of filters.

$$x_1 \approx \ldots \approx x_k \prec y_1 \approx \ldots \approx y_l \tag{4}$$
$$x_1^1 \approx \ldots \approx x_{l_1}^1 \prec \ldots \prec x_1^n \approx \ldots \approx x_{l_n}^n \tag{5}$$

If $\mathcal{F}(\Gamma)$ contains a filter than is not of the form (2) and a filter that is not of the form (3), then one of the following cases holds.

1. The language Γ contains a filter of the form (4) where $k > 1$ and $l \geq 1$ as well as a filter of the same form where $l > 1$ and $k \geq 1$, or
2. there exists a filter of the form (5) for $n \geq 3$.

The next two sections handles these cases. The section 6.1 covers the first situation. The second case is taken care of by the section 6.2.

6.1 Filters Of Range 2

Here we prove the following.

Proposition 1. *Let Γ be a positive temporal template with filters $\preceq_L$ and $\preceq_R$ defined as follows. The preorder $\preceq_L$ is of the form (4) with $k > 1$ and $l \geq 1$. The preorder $\preceq_R$ is also of the same form (4), but with $l > 1$ and $k \geq 1$. Then sPol(Γ) contains only essentially unary polymorphisms.*

Our strategy here is to show first that 'short' relations with 'short' filters of the form (4) with $k > 1$ and $l \geq 1$ are closed only under surjective unary polymorphisms or they express $(x_1 \leq x_2 \vee x_1 \leq x_3)$. Afterward, we consider arbitrary relations with arbitrary 'long' filters. For a 'long' relation R_L, we show that it can express a 'short' relation R_S. Therefore R_L can express everything expressible by R_S; equivalently, R_L cannot have more surjective polymorphisms than R_S. Lemmas 5, 4, and 9 ensure that R_S obtained from R_L has an appropriate filter. 'Short' relations have either arity 3 or 4. The first case is handled by Corollary 5, the second case by Lemma 17. See the example at the end of this subsection. First, we give two preliminary lemmas.

Lemma 15. *Let $R(x_1, x_2, x_3)$ be a positive temporal relation with a filter $x_1 \approx x_2 \prec x_3$. Moreover assume that $x_3 \prec x_1 \approx x_2$ does not belong to $\mathcal{F}(R)$. Then $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$.*

Lemma 16. *Let $R(x_1, x_2, x_3)$ be a positive temporal relation with filters $x_1 \approx x_2 \prec x_3$ and $x_3 \prec x_1 \approx x_2$. Then sPol$(\Gamma)$ contains only essentially unary polymorphisms.*

As an immediate consequence of lemmas 15 and 16 we get the following.

Corollary 5. *Let $R(x_1, x_2, x_3)$ be a positive temporal relation with a filter $x_1 \approx x_2 \prec x_3$. Then either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [\Gamma]$ or sPol(Γ) contains only essentially unary polymorphisms.*

Lemma 17. *Let $R(x_1, x_2, x_3, x_4)$ be a positive temporal relation with a filter $x_1 \approx x_2 \approx x_3 \prec x_4$. Then either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$ or sPol(R) contains only essentially unary polymorphisms.*

Lemma 18. *Let $R(x_1, \ldots, x_n)$ be a positive temporal relation. If R has any filter of the form (4) where $k > 1$ and $l \geq 1$, then either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$ or sPol(Γ) contains only essentially unary polymorphisms.*

Similarly, we can show that if a positive temporal relation has any filter of the form (4) where $k \geq 1$ and $l > 1$, then either it is closed only under essentially unary surjective polymorphisms or it can express $(x_2 \leq x_1 \vee x_3 \leq x_1)$. The following statement in fact ends the proof of Proposition 1.

Lemma 19. *If both $(v_1 \leq v_2 \vee v_1 \leq v_3)$ and $(v_2 \leq v_1 \vee v_3 \leq v_1)$ belong to $[\Gamma]$, then Γ is closed only under essentially unary polymorphisms.*

Proof of Proposition 1. Let R be a relation of Γ with $\preceq_L$. Then, by Lemma 18, either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$ or *sPol*(R) contains only essentially unary polymorphisms. Similarly, for some R_1 that has $\preceq_R$ as a filter we can show that either all its surjective polymorphisms are essentially unary or $(v_2 \leq v_1 \vee v_3 \leq v_1)$ belongs to $[R_1]$. To complete the proof we use Lemma 19. □

We finish this subsection with an example that illustrates Proposition 1.

Example 4. Consider a positive relation R_L given by $(x_1 \leq x_2 \vee x_1 \leq x_3) \wedge (x_5 \leq x_4 \vee x_6 \leq x_4) \wedge \phi(y_1, \ldots, y_m)$ where $\{x_1, \ldots, x_6\} \cap \{y_1, \ldots, y_m\} = \emptyset$. It is straightforward to show that $x_2 \approx x_3 \prec x_1$ as well as $x_4 \prec x_5 \approx x_6$ are minimal wrt $\ll$ filters of R –

see Example 3. We now claim that *sPol*(Γ) contains essentially unary polymorphisms only or equivalently, by Corollary 3, the problem *QCSP*(Γ) is NP-hard.

Now, define R_S as $\exists y_1 \ldots \exists y_m \exists x_4 \exists x_5 \exists x_6 R_L(x_1, \ldots, x_6, y_1, \ldots, y_m)$ By Lemma 5, we have that R_S inherits the filter $x_2 \approx x_3 \prec x_1$. From Lemma 15 we infer that $(v_1 \leq v_2 \vee v_1 \leq v_3)$ belongs to $[R_S]$. Hence it belongs to $[R_L]$; note that R_S is cp-definable in R_L – recall the definition of R_S above. Similarly we can show that $(v_2 \leq v_1 \vee v_3 \leq v_1) \in R_L$. By Lemma 19 we have that *sPol*(R_L) contains essentially unary polymorphisms only.

6.2 Filters Of Range Greater Than 2

What remains to prove is the following.

Proposition 2. *Let Γ be a positive temporal language. If there exists any minimal with respect to $\ll$ filter in $\mathcal{F}(\Gamma)$ whose range is strictly greater than 2, then sPol(Γ) contains only essentially unary polymorphisms.*

The strategy here is similar to one in the preceding section. We express 'short' relations with 'short' filters using 'long' relations with 'long' filters and show that 'short' relations are closed under essentially unary surjections only. Here, 'short' filters are of the form (5) where each $l_i = 1$ for all $1 \leq i \leq n$. In turn, we think that a relation R_S is 'short' if it contains a 'short' filter $\preceq_S$ and $Dom(R_S) = Dom(\preceq_S)$. Each 'long' relation has at least one (arbitrary) filter of the form (5). For example, the filter $x_1 \approx x_2 \prec x_3 \prec x_4$ is 'long', but the filter $x_1 \prec x_3 \prec x_4$ is 'short'.

7 Proof of Theorem 2

In [3] it is shown that each positive temporal language Γ from case 2 is decidable in NLOGSPACE. To prove Theorem 1 we need hardness as well.

Lemma 20. *Let Γ be a positive temporal language such that each its relation is definable as a conjunction of weak inequalities but not as a conjunction of equalities. Then QCSP(Γ) is NLOGSPACE-complete.*

Proof. (of Theorem 2) By Theorem 5 we have that each positive temporal Γ is either definable as in one of the conditions 1–4 of Theorem 1 or it is closed under essentially unary surjections only. Lemma 20 gives us the complexity characterization of item 2. Item 1 is characterized in [4]. In [10] we give the complexity proof for items 3 and 4. Finally, by Corollary 3, we have that for any other positive temporal language Γ the problem *QCSP*(Γ) is NP-hard. □

Acknowledgements We thank Jerzy Marcinkowski for turning our attention to [4].

References

1. James F. Allen. Maintaining knowledge about temporal intervals. *Commun. ACM*, 26(11):832–843, 1983.
2. Manuel Bodirsky. *Constraint Satisfaction Problems with Infinite Domains*. PhD thesis, Humboldt-Universität zu Berlin, 2004. `http://www2.informatik.hu-berlin.de/~bodirsky/publications/diss.html`.
3. Manuel Bodirsky and Hubie Chen. Qualitative temporal and spatial reasoning revisited. In *CSL 2007, Annual Conference of the EACSL, Proceedings*, LNCS. Springer.
4. Manuel Bodirsky and Hubie Chen. Quantified equality constraints. In *22nd IEEE Symposium on Logic in Computer Science (LICS 2007), Proceedings*. IEEE Computer Society.
5. Manuel Bodirsky and Jan Kára. A fast algorithm and lower bound for temporal reasonning. `http://www2.informatik.hu-berlin.de/~bodirsky/en/publications.php`.
6. Manuel Bodirsky and Jan Kára. The complexity of equality constraint languages. In *Computer Science - Theory and Applications, First International Computer Science Symposium in Russia, CSR 2006, Proceedings*, LNCS, pages 114–126. Springer, 2006.
7. Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. In Richard E. Ladner and Cynthia Dwork, editors, *Proceedings of the 40th Annual ACM Symposium on Theory of Computing*, pages 29–38. ACM, 2008.
8. F. Boerner, A. Bulatov, P. Jeavons, and A. Krokhin. Quantified constraints: Algorithms and complexity. In *Proceedings of CSL and the 8th Kurt Gödel Colloquium*, LNCS. Springer.
9. A.A. Bulatov. A dichotomy theorem for constraints on a three-element set. In *Proceedings 43rd IEEE Symposium on Foundations of Computer Science (FOCS'02)*.
10. Witold Charatonik and Michał Wrona. Quantified positive temporal constraints. In *CSL 2008, Annual Conference of the EACSL, Proceedings*, LNCS, pages 94–108. Springer.
11. David Cohen and Peter Jeavons. The complexity of constraints languages. In Francesca Rossi, Peter van Beek, and Toby Walsh, editors, *Handbook of Constraint Programming*. Elsevier, 2006.
12. T. Feder and M.Y. Vardi. Monotone monadic SNP and constraint satisfaction. In *Proceedings of 25th ACM Symposium on the Theory of Computing (STOC)*, pages 612–622, 1993.
13. M. Fisher, D.Gabbay, and L. Vila, editors. *Handbook of Temporal Reasoning in Artificial Intelligence*. Elsevier, 2005.
14. Wilfried Hodges. *A shorter model theory*. Cambridge University Press, 1997.
15. P.G. Jeavons, D.A. Cohen, and M. Gyssens. Closure properties of constraints. *Journal of the ACM*, 44:527–548, 1997.
16. A. Krokhin, P. Jeavons, and P. Jonsson. A complete classification of complexity in Allens algebra in the presence of a non-trivial basic relation. In *Proceedings of the 17th International Joint Conference on Artificial Intelligence*, pages 83–88. Morgan Kaufmann, 2001.
17. Rolf H. Möhring, Martin Skutella, and Frederik Stork. Scheduling with and/or precedence constraints. *SIAM J. Comput.*, 33(2):393–415, 2004.
18. J. Renz and B.Nebel. Qualitative spatial reasoning using constraint calculi. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of Spatial Logics*. Springer Verlag, 2007.
19. T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings 10th ACM Symposium on Theory of Computing, STOC'78*, pages 216–226, 1978.