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Złożoność kwantyfikowanych problemów spełniania więzów dla pozytywnych języków temporalnych.

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A Complete Complexity Characterization of Quantified Positive Temporal Constraints

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Contents

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Abstract

Constraint satisfaction problems (*csp*) constitute a coherent approach to research on a wide selection of combinatorial problems. The computational problem *CSP*(Γ, D) for a set of relations Γ over domain D is the question whether a given primitive positive formula ψ with all its variables existentially quantified is true in Γ. A *quantified constraint satisfaction problem* (*qcsp*) is a version of a constraint satisfaction problem where variables occurring in ψ can be not only existentially but also universally quantified.

We say that a relation is *temporal positive* if it has a positive first order definition over the order of rational numbers. This dissertation concerns quantified constraint satisfaction problems for finite sets (*languages*) of positive temporal relations. We give a complete complexity characterization of such problems in the sense that we describe the complexity of *QCSP*$(\Gamma, \mathbb{Q})$ for all positive temporal languages Γ where $\mathbb{Q}$ denotes the set of rational numbers.

The complexity of these problems varies. Some of them are in *LOGSPACE*, some are *NLOGSPACE-complete*, *P-complete*, *NP-complete*, or *PSPACE-complete*. Hence under the assumption that *LOGSPACE* $\subsetneq$ *NLOGSPACE* $\subsetneq$ *P* $\subsetneq$ *NP* $\subsetneq$ *PSPACE* we obtain a hierarchy of the complexity of quantified constraint satisfaction problems for positive temporal languages. We also give a simple logical and a simple algebraical criteria distinguishing the levels of this hierarchy.

In order to provide our characterization we use so called *algebraical approach* to constraint satisfaction problems based on the notion of a *polymorphism*. A polymorphism is a function that preserves a relation. Main tools in this framework are *preservation theorems* that bind functions with some variant of logical definability. In particular, in the case of quantified constraint satisfaction problems the *surjective* preservation theorem concerns *surjective* functions. Roughly speaking, it states that the more surjections a relation preserve the lower is its expressive power and the complexity of its qcsp. Qcsps for languages preserved by the same surjective functions are equivalent with respect to reducibility in logarithmic space. To provide our complexity hierarchy we first characterize positive temporal languages according to surjective polymorphisms (we show that there are only nine families of positive temporal relations closed under different surjective functions on $\mathbb{Q}$). Then we give their syntactical characterization in terms of first order logic, and finally we arm each such family with the complexity of their quantified constraint satisfaction problems.

Apart from algebraical and logical methods, which are quite well known, we provide new methods. What should be mentioned here is our *filter representation* of positive temporal relations that enables us to use powerful algebraical and logical tools in a relatively simple way. In particular, filters of a temporal relation imply a simple

syntactical definition of the relation in terms of first order logic and facilitate using the surjective preservation theorem.

Chapter 1

Introduction

1.1 Constraint Satisfaction Problems

We start by introducing a notion of a *constraint satisfaction problem*. Let $\langle \Gamma, D \rangle$ be a *relational structure* (called also a *constraint language* or *template*) with the set of relations Γ over some domain D, and of some signature τ. A constraint satisfaction problem for Γ, denoted also by *CSP*(Γ, D) or just *CSP*(Γ), is a computational problem that asks whether a *pp-formula* (*positive primitive formula*), which has the form:

$$R_1(x_1^1, \ldots, x_{m_1}^1) \wedge \ldots \wedge R_n(x_1^n, \ldots, x_{m_n}^n), \tag{1.1}$$

is satisfiable in Γ. Each relational symbol R_i from (1.1) is interpreted as one of the relations from Γ.

The research on constraint satisfaction problems go back more than thirty years ago [35, 31]. Combinatorial programs that can be formulated as csps (constraint satisfaction problems) come from as different areas of computer science as picture processing [35], boolean satisfiability [40, 20], database theory [41], graph colouring [2, 25], temporal reasoning [7, 11] and many others.

In general, in this thesis we study constraint satisfaction problems over infinite domains. Although csps with finite domains [19] are better known, there is a quite recently proposed and quickly developing approach for infinite domains. It was provided by Manuel Bodirsky [4, 3] and his co-authors.

Finite domain constraints have their own famous Feder-Vardi Dichotomy Conjecture [21], complexity classifications like Schaefer's [40] and Bulatov's [14] theorems and the algebraic approach discovered by Jeavons, Cohen, and Gyssens [28].

The Bodirsky's framework to infinite domain constraints, which focuses on ω-categorical relational structures, inherits the algebraic approach from the finite domain case and already have some complexity characterizations theorems proved by Bodirsky, Kára and Chen [10, 5, 11]. Likewise in the finite case, constraint satisfaction problems with infinite domains may constitute a good framework for some problems in artificial intelligence, see for example [7]. Let us just mention here that the complexity classification theorem from [11] can provide a powerful tool in temporal reasoning.

We now give two examples of constraint satisfaction problems with infinite domains. They are taken from [4].

Example 1. *Let us consider the* DIGRAPH ACYCLITY *problem. The instance of this problem is a digraph* $D = \langle V, E \rangle$ *and it is asked whether there is no directed cycle* *in D.*

Let us note that DIGRAPH ACYCLITY problem is equivalent to CSP$(\langle <, \mathbb{Q} \rangle)$ where $\mathbb{Q}$ is the set of rational numbers and $<$ is interpreted as a dense linear order over rationals. It is not hard to prove that a digraph can be mapped to a linear order if and only if it is acyclic.

The problem CSP$(\langle <, \mathbb{Q} \rangle)$ may be solved in polynomial time [44].

Example 2. *The second example we will consider is the NP-complete problem* BETWENNESS. *We are given a finite set V and a collection of ordered triples (x,y,z) of different elements from V. We ask whether there is a one-to-one function $f : V \to \{1, \ldots, |V|\}$ such that, for each (x, y, z), we have either $f(x) < f(y) < f(z)$ or $f(z) < f(y) < f(x)$.*

Let us note that BETWEENNESS can be modeled by CSP$(B, \mathbb{Q})$ where $B(x, y, z) = \{(x, y, z) \in \mathbb{Q}^3 | x < y < z \vee x > y > z\}$. A complexity proof for BETWEENNESS can be found in [23].

Constraint satisfaction problems that model DIGRAPH ACYCLITY and BETWEENNESS are special cases of csps for relations that are first order definable over the order of rational numbers. The complexity of such relations was classified in [11].

1.2 Quantified Constraint Satisfaction Problems

Again we have a set of relations Γ over some domain and of some signature τ, but this time we ask whether a *quantified conjunctive formula* (*qc-formula*) is true in Γ.A qc-formula is a formula of the form:

$$Q_1 x_1 \ldots Q_k x_k \bigwedge_{i=1}^{n} R_i(v_1^i, \ldots, v_{m_i}^i), \tag{1.2}$$

where $Q_i \in \{\forall, \exists\}$, $R_i \in \tau$ for all $1 \leq i \leq n$, m_i is the arity of R_i, and $v_j^i \in \{x_1, \ldots, x_n\}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$.

The set of qc-formulas over τ that are true in Γ is called a *quantified constraint satisfaction problem* for Γ and is denoted by *QCSP*(Γ, D), or by *QCSP*(Γ) if a domain is known from the context.

Let us note that there is an obvious one-to-one correspondence between pp-formulas satisfiable in some Γ and the set of existentially quantified qc-formulas true in Γ. Therefore, in this sense we can see quantified constraint satisfaction problems (qcsps) as a generalization of constraint satisfaction problems. Moreover, sometimes by saying: constraint satisfaction problems, we do not mean computational problems but the whole field of research including also qcsps and other variations of csps. It will be clear from the context which meaning of csps we have in mind.

There is a useful game-based approach for quantified constraint satisfaction problems that often helps to find the complexity of such problems. Let us now consider a qc-formula $\psi := Q\phi$ with a quantifier prefix Q and a quantifier-free part ϕ. We want to check whether ψ is true in some relational structure Γ over some domain D. As it is usual, we see ψ as a two-player game between the existential and the universal player. They are playing in the order prescribed by the quantifier prefix. If the leftmost quantifier is existential (universal) and it binds a variable x, then the existential (universal) player begins and she (he) sets to x some value from D. Then the second leftmost variable is evaluated and so forth. The existential (universal) player wins if the valuation of the variables obtained during the game satisfies (does not satisfy) ϕ. It is widely known that ψ is true in Γ if and only if the existential player has a winning strategy, that is, she wins in each game based on ψ independently of the moves of her universal opponent. Otherwise, ψ is false and the universal player has the winning strategy.

If a variable is bound by an existential (a universal) quantifier than we call it existential (universal). We say that x is earlier (later) than y if it occurs earlier (later) in the quantifier prefix than y.

In the next example we show how to check whether a given finite automaton generates an empty language using some quantified constraint satisfaction problem. We consider only automata whose initial state does not belong to the set of final states. For the formal definitions of a finite automaton and corresponding notions we refer the reader to [27].

Example 3. *Let QCSP*$(\{x_1 \geq x_2\})$ *be a quantified constraint satisfaction problem for a relation* $\{(x_1, x_2) \in \mathbb{Q}^2 \mid x_1 \geq x_2\}$ *and let* $\mathbb{A} = \langle \mathcal{A}, \mathcal{Q}, q_0, \delta, \mathcal{F} \rangle$ *where* $\mathcal{A}$ *is an alphabet,* $\mathcal{Q}$ *is a set of states,* q_0 *is an initial state,* $\delta \subseteq \mathcal{Q} \times \mathcal{A} \to \mathcal{Q}$ *is a transition function, and* $\mathcal{F} = \{q_{f_1}, \ldots, q_{f_{|F|}}\} \subseteq \mathcal{Q}$ *is a set of accepting states. We also assume that* $q_0 \notin \mathcal{F}$.

For each automaton $\mathbb{A}$ *as above we construct an instance of QCSP*$(\{x_1 \leq x_2\})$ *of the following form:*

$$\exists x_{q_0} \forall x_{q_{f_1}} \ldots \forall x_{q_{f_{|F|}}} \exists x_{q_{r_1}} \ldots \exists x_{q_{r_{|Q|-|F|-1}}} \bigwedge_{(q_1,a,q_2) \in \delta} x_{q_1} \geq x_{q_2} \tag{1.3}$$

Note that if $\mathbb{A}$ *generates a non-empty language, then in the above formula the value of* x_{q_0} *has to be greater than or equal to the value of each* x_{q_f} *such that* q_f *is a final state and it accepts some word from* $\mathcal{A}^*$. *Note that each* x_{q_f} *such that* $q_f \in \mathcal{F}$ *is universal and later than* x_{q_0}. *Therefore the universal player can easily win by setting each* x_{q_f} *to the value strictly greater than the value of* x_{q_0}.

On the other hand, a quantifier-free part of the formula (1.3) implies $x_{q_1} \geq x_{q_2}$ *for some* $q_1, q_2 \in \mathcal{Q}$ *if and only if* q_2 *is reachable from* q_2 *via some word* $w \in \Sigma^*$. *Because the universal player cannot falsify anything except* $x_{q_0} \geq x_{q_f}$ *for some* $q_f \in \mathcal{F}$, *he cannot win if no accepting state is reachable from* q_0, *or equivalently if* $\mathbb{A}$ *generates the empty language.*

In Section 4.1 we show that QCSP$(\{x_1 \geq x_2 \vee x_1 \geq x_3\})$ *solves the nonemptiness problem for context-free grammars.*

Both *QCSP*$(\{x_1 \geq x_2\})$ and *QCSP*$(\{x_1 \geq x_2 \vee x_1 \geq x_3\})$ are the instances of quantified problems for positive temporal languages, which are the subject of this thesis.

The examples of quantified constraints satisfaction problems that model other natural problems can be found in [33, 39, 24]

Since the whole thesis is about (quantified) constraint satisfaction problems, as we suppose, we owe the reader a short explanation for why we find csps important and interesting to study. The properties of constraints problems which we find essential are shortly discussed in the following sections of this chapter.

1.3 Feder-Vardi Dichotomy Conjecture

Undisputedly, one of the most interesting problems in the area of csps is to verify the Dichotomy Conjecture posed by Feder and Vardi [21]. It says that every constraint satisfaction problem over a finite domain is either polynomial or NP-complete. This conjecture was inspired by Schaefer's Dichotomy Theorem for csps on a two element set [40]. Thanks to Ladner [29], we know that if P is not equal to NP, then there is an infinite number of non-empty complexity classes between P and NP. Ladner's intermediate problems are created using diagonal method and hence are artificial. Therefore it makes sense to look for a way to describe natural problems in NP. Unfortunately, the csps framework for finite domains does not capture all such problems [21, 32]. However, there is a wide selection of problems that are csps in nature and it would be interesting anyway to know whether at least this subset of nondeterministic polynomial problems exhibit dichotomy.

Up till now, this conjecture was verified in some special cases only. Probably the most important of them is a dichotomy for relations over three element domain proved by Bulatov [14] and a dichotomy for templates that are digraphs with no sources and no sinks [2]. What should be mentioned here is that looking for dichotomy in csps we can restrict ourselves to templates that are digraphs. It was proved in [21] that each csp has a polynomially equivalent digraph homomorphism problem.

In general we could expect that all constraint satisfaction problems and their variations like quantified csps are natural in the sense that they are complete for natural complexity classes. Unfortunately it is not true. In particular each computational problem has a polynomially equivalent constraint satisfaction problem over infinite domain [8]. Even among ω-categorical templates there are known those that, under the assumption that P is not equal to co-NP, belong to co-NP but are not polynomial [8].

However, it is expected that finite domain csps exhibit dichotomy. Moreover, there are also some important families of infinite csps like equality [10] and temporal [11] constraints that are naturally splitted into polynomial and NP-complete subsets. What should be mentioned is that there are similar complexity classifications for quantified constraints [20, 18, 5]. For example, it is known that each quantified constraint satisfaction problem over two element domain belongs to P or is PSPACE-complete [20].

1.4 (Quantified) Complete Classifaction Theorems and Meta-Problem

In the previous section we mentioned a problem of verifying the conjecture of Feder and Vardi. If we classified the complexity of all csps over finite domain, we would verify the conjecture. In general we might be interested in the following problem parametrized by a (possibly infinite) family of relations.

Research Problem 1. *Let Γ be an infinite family of relations over some domain. Establish the complexity of CSP(Γ') for each (in this thesis finite) subset Γ' of Γ.*

In a similar way, we can formulate a research problem concerning quantified or other variations of csps. A theorem that solves Research Problem 1 is called a *complete classification theorem*. The difference between csps for finite and infinite constraint languages will be discussed shortly in Section 2.1.

In complexity theory it is common to look for a simple problem by somehow restricting a hard one. Likewise, each complete classification theorem concerns some hard problem. For instance, the Schafer's theorem concerns the boolean satisfiability, which is widely known to be NP-complete and therefore computationally hard. Nevertheless some special cases like Horn-SAT or 2-SAT are polynomially tractable. What might be interesting is that all tractable subcases of boolean satisfiability known long before Schafer's classification can be also obtained by restricting the constraint language Γ where Γ is a set of all boolean relations. Therefore in this case csps gives a natural framework to look for simple restrictions of hard problems.

Temporal and spatial reasoning [38, 22] is a subdiscipline of artificial intelligence that concerns reasoning about temporal dependencies between events and spatial regions. There is a plenty of formalisms based on csps over infinite domains that are used in this area. Among others there are point algebra, Ord-Horn constraints [36], AND/OR precedence constraints [34] and Allen's interval algebra [1]. As it was argued in the series of papers [4, 11, 7, 9] all of these formalisms may be seen as csps for constraint languages of relations first order definable over $\langle Q, < \rangle$. Such relations are called *temporal relations*. Likewise Schafer's approach to boolean satisfiability, this uniform approach to research on temporal constraints led to complete classification given in [11], but this time the classification revealed new, not known before tractable classes of constraints.

All complexity classification theorems known to the author have one more good feature. There is an algorithm that given a finite subset Γ' of considered family Γ can decide what is the complexity of *CSP*(Γ'). A problem whether such an algorithm exists is called the *meta-problem* — see for example [20]. In some sense the complexity of the meta-problem reveals the structure of the considered family of relations.

Research Problem 2. *Suppose that a complete classification theorem for a family of relations Γ is known. Establish the complexity of the following problem.*
Given: A (in this thesis, finite) subset Γ' of Γ.
Question: What is the complexity of CSP(Γ')?

Complete classification theorems were discussed many times before. Probably more wide-ranging discussions about classifying the complexity of csps can be found in [40, 20, 18].

We now give an example of a complete classification theorem for quantified constraint satisfaction problems over infinite domains. We consider a set of relations that are first order definable over $\langle \mathbb{N}, \emptyset \rangle$, that is, our domain is a set of natural numbers (or any other countable domain) and the only relational symbol is equality. Finite sets of such relations are called *equality languages*. Recall that $Th(\langle \mathbb{N}, \emptyset \rangle)$ admits quantifier elimination. It allows us to consider only quantifier-free formulas in the following theorem [5].

Theorem 4. *Let Γ' be an equality language. Then exactly one of the following holds.*

1. ***Negative languages**. Relations of such a language are definable as CNF-formulas whose clauses are either equalities ($x = y$) or disjunctions of disequalities ($x_1 \neq y_1 \vee \ldots \vee x_k \neq y_k$). For each negative Γ the problem QCSP(Γ) is contained in LOGSPACE.*

2. ***Positive languages**. Relations may be defined as a conjunction of disjunctions of equalities ($x_1 = y_1 \vee \ldots \vee x_k = y_k$). For each positive Γ not being negative the problem QCSP(Γ) is NP-complete.*

3. *In all other cases the set QCSP(Γ) is PSPACE-complete.*

As we see, this theorem gives us a straightforward way to distinguish between simple and hard quantified constraint satisfaction problems for equality languages.

It should be noted that this characterization makes sense under the assumption that *LOGSPACE* $\subsetneq$ *NP* $\subsetneq$ *PSPACE*, which is believed to hold. What about the meta-problem? If relations are given as formulas of the appropriate form, then it is obviously decidable. Another way of representing ω-categorical templates and the meta-problem for this representation will be discussed later.

It should be noted that $Th(\mathbb{N}, \emptyset)$ is the simplest ω-categorical theory in the sense that characterizing complexity of (q)csps for relations of an arbitry ω-categorical theory requires characterizing the complexity of relations definable in $Th(\mathbb{N}, \emptyset)$.

1.5 Our Main Contribution as an Example of Complete Classification Theorem

In this section we classify the complexity of quantified constraint satisfaction problems for *positive temporal relations*. We defined temporal relations as those that can be first order defined over the order of rational numbers. Likewise, a positive temporal relation has a positive definition over $\langle Q, \leq \rangle$, that is, to build positive temporal formulas we use $\leq$ interpreted as the weak order over rational numbers, conjunction and disjunction. Roughly speaking, we do not consider negation. However, the complexity of the simplest qcsps over positive temporal languages are in LOGSPACE whereas the hardest such problems are PSPACE-complete. As usual in this thesis, a *positive temporal language* is a finite set of positive temporal relations.

Example 5. *The set of positive temporal languages varies with respect to the complexity of qscps. Among others it contains the language* $\{x_1 \leq x_2\}$ *introduced in Section 1.2. As we will see, QCSP*$(\{x_1 \leq x_2\})$ *is NLOGSPACE-complete. Further, the problem QCSP*$(\{(x_1 \leq x_2 \vee x_1 \leq x_3)\})$ *models a quantified positive variation of AND/OR precedence constraints, which are used in scheduling [34]. The set QCSP*$(\{(x_1 \leq x_2 \vee x_1 \leq x_3)\})$ *is P-complete. Let us also mention QCSP*$(\{(x_1 \leq x_2 \vee x_2 \leq x_3)\})$*. This one is PSPACE-hard.*

The complete complexity classification of quantified constraint satisfaction problems for positive temporal languages is the main contribution of this thesis and is given by the following theorem. Originally the proof of this theorem was published in [16] and in [17].

Theorem 6. ***The Main Theorem*** *Let* Γ *be a language of positive temporal relations, then one of the following holds.*

1. *Each relation in* Γ *is definable by a conjunction of equations* ($x_1 = x_2$) *and then QCSP*(Γ) *is in LOGSPACE.*

2. *Each relation in* Γ *is definable by a conjunction of weak inequalities* ($x_1 \leq x_2$) *and then QCSP*(Γ) *is in NLOGSPACE.*

3. *Each relation in* Γ *is definable by a formula of the form* $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ *and then, provided* Γ *satisfies neither condition 1 nor 2, the set QCSP*(Γ) *is P-complete.*

4. *Each relation in* Γ *is definable by a formula of the form* $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$ *and then, provided* Γ *satisfies neither condition 1 nor 2, the set QCSP*(Γ) *is P-complete.*

5. *Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} = y_{i_1} \vee \ldots \vee x_{i_k} = y_{i_k})$ and then, provided Γ does not belong to any of the classes 1–4, the set QCSP(Γ) is NP-complete.*

6. *The problem QCSP(Γ) is PSPACE-complete.*

We want to make here a similar remark as in the previous chapter. It seems that our characterization can be useful and possibly interesting only under the assumption that $L \subsetneq NL \subsetneq P \subsetneq NP \subsetneq PSPACE$. In fact, all such characterizations are provided under similar complexity theoretical assumptions.

Theorem 6 solves a kind of Research Problem 1 where Γ is a set of positive temporal relations and we consider quantified csps instead of csps. It is natural to ask about Research Problem 2 that is about the complexity of a meta-problem. For the moment we make a similar remark as in the case of Theorem 4 in the previous section. If relations are given in the specific clausal form preferred by the preceding formulation of the theorem, then a meta-problem is obviously linear. A meta-problem for a different representation will be discussed later.

Our contribution concerns two lines of research. On one hand we classify quantified constraint satisfaction problems [20, 12, 18, 5] but on the other hand the problems we classify are temporal and therefore we also touch the topic of temporal reasoning [38, 22, 36, 34, 1, 4, 11, 7, 9].

In a very recent paper [11] the authors give a classification of constraint satisfaction problems over temporal languages. Although it sounds similar, it is substantially different from the classification of quantified constraint satisfaction problems for positive temporal languages in this thesis.

What should be noted here is that some partial results for Theorem 6, like characterizations of cases 1, and 5 come directly from Theorem 4, which was proved in [5]. The characterization of case 2 is taken from [7]. Theorem 6 substantially improves these results in the sense that we consider a strictly more expressive class of constraint languages. When it comes to details we will see that in several places we use methods similar to those from [5].

Chapter 2

Preliminaries

This chapter is dedicated to introduce the necessary definitions. In most cases we follow the notation from [4, 5, 26].

2.1 Quantified Constraints Satisfaction Problems

Let τ be some relational (in this thesis always finite) signature, i.e., a set of relational symbols with assigned arity. Then Γ is a τ-structure over domain D if for each relational symbol R_i from τ, it contains a relation of according arity defined on D. In the rest of the thesis we usually say *constraint language* (or *template*) instead of relational structure. Moreover, we use the same notation for relational symbols and relations.

We also identify relations with formulas defining these relations in the following sense. If $\phi(x_1, \ldots, x_n)$ is a formula with free variables $x_1, \ldots, x_n$ then we identify the formula ϕ with the relation $\{\langle x_1, \ldots, x_n\rangle \mid \phi(x_1, \ldots, x_n)\}$. The order of variables in ϕ is usually left implicit. Note that with this convention $x_1 \leq x_2$ and $v_1 \leq v_2$ are two different notations for the same relation, which is different from the relation $x_2 \leq x_1$. Sometimes we write $R(x_1, \ldots, x_n)$ to emphasize the arity and the order of variables in R. Moreover, a formula of the form $x_1 \leq x_2 \wedge \ldots \wedge x_{n-1} \leq x_n$ is often abbreviated to $x_1 \leq \ldots \leq x_n$.

This thesis is about quantified constraint satisfaction problems. Each quantified constraint satisfaction problem is parametrized by a (in this thesis, finite) set of relations Γ over some domain D. Each instance of qcsps for Γ contains a scope, that is, a set of variables. Tuples of variables are constrained by relations over D. Each variable is either existentially or universally quantified. The quantification and the dependencies between variables in the scope are expressed by a *quantified conjunctive formula* (qc-formula).

Definition 7. *Let Γ be a relational τ-structure containing $R_1, \ldots, R_k$. Then a quantified conjunctive formula over Γ is a formula of the following form:*

$$Q_1 x_1 \ldots Q_n x_n (R_1(\vec{v_1}) \wedge \ldots \wedge R_k(\vec{v_k})), \tag{2.1}$$

where $Q_i \in \{\forall, \exists\}$ and $\vec{v_j}$ are vectors of variables $x_1, \ldots, x_n$.

We are now ready to give a formal definition of a quantified constraint satisfaction problem.

Definition 8. *Let Γ be a τ-structure over a domain D. The QCSP(Γ, D) (more often denoted QCSP(Γ)) is the following computational problem.*

Instance: *A quantified conjunctive formula ψ without free variables over τ.*

Question: *Is ψ true in Γ over D?*

Because τ is finite we can assume that the size of the instance is just a size of the qc-formula and does not depend on the size of the representation of relations.

Let us consider a constraint language Γ. As it appears, we can in some way define and then add defined relations to Γ without affecting the complexity of its qcsp. First we introduce an appropriate notion of defining and then we say which relations can be safely added.

Definition 9. *A relation R has a* qc-definition *in a language Γ if there exists a qc-formula $\phi(x_1, \ldots, x_n)$ over Γ such that for all $a_1, \ldots, a_n$ we have $R(a_1, \ldots, a_n)$ iff $\phi(a_1, \ldots, a_n)$ is true. The set of all relations qc-definable in Γ is denoted by* $[\Gamma]$.

The next lemma says that arbitrary relations from $[\Gamma]$ can be added to Γ without substantially changing the complexity of its qcsp.

Lemma 10 ([5])**.** *Let Γ_1, Γ_2 be relational languages. If every relation in Γ_1 has a qc-definition in Γ_2, then QCSP(Γ_1) is log-space reducible to QCSP(Γ_2).*

Global and local tractability. In this thesis, as it is common, we consider the complexity of (q)csps for *finite* positive temporal languages. However, one can ask about an infinite template Γ_∞. The natural approach (*global tractability*) is that in the instance of the problem we allow arbitrary relations from Γ_∞. Nevertheless in this case the instance has also to contain definitions of these relations. (If there are infinitely many relations we cannot assume that their definition is finite.) The problem is that global tractability does not fit to algebraic approach as well as *local tractability*. We say that Γ_∞ is locally tractable if all its finite subsets are tractable. In general we may say that in this approach the complexity of (q)csps of some infinite language is the complexity of its hardest finite subset. The belief that local and global tractability coincide is known as the *local-global conjecture*.

2.2 Logic and Model Theory

The definition of a quantified constraint satisfaction problem in the previous section allows arbitrary templates. As it was noted before, this thesis is about relations definable in the theory of the order of rational numbers ($Th(\mathbb{Q}, <)$). Due to the theorem of Cantor, it is widely known that all countable models of this theory are isomorphic. Theories whose all countable models are isomorphic are called ω-categorical. Likewise, a structure (in this case a constraint language) is called ω-categorical if its theory is ω-categorical.

In comparison to arbitrary infinite templates, the ω-categorical ones inherit some properties of finite constraint languages. One of them is a corollary of the theorem of Engeler, Ryll-Nardzewski and Svenonius. Before we give it we need to define some more notions.

A vector of elements $\langle t_1, \ldots, t_k \rangle$ is often denoted by $\vec{t}$. Automorphisms of a relational structure Γ constitute a group with respect to composition. An orbit of a k-tuple $\langle t_1, \ldots, t_k \rangle$ in Γ is the set of tuples $O_{\langle t_1, \ldots, t_k \rangle} = \{\langle \Pi(t_1), \ldots, \Pi(t_k) \rangle \mid \Pi \in Aut(\Gamma)\}$.

We say that a group of automorphisms of Γ is *oligomorphic* if for each k the set $\{O_{\langle t_1,\ldots,t_k\rangle} \mid t_1,\ldots,t_k \in D\}$ is finite.

Theorem 11. ***(Engeler, Ryll-Nardzewski, Svenonius)*** *Let T be a complete theory, with at least one infinite model, in some countable first order language L. Then the following are equivalent.*

1. *T is ω-categorical.*
2. *A group of automorphisms of each countable model of T is oligomorphic.*
3. *For each finite set of variables $\{x_1,\ldots,x_k\}$, there are only finitely many pairwise non-equivalent formulas $\phi(x_1,\ldots,x_k)$ of L modulo T.*

Note that Theorem 11 suggests two ways of representing ω-categorical relations of finite arity. First of them, based on formulas, was used in the previous chapter to formulate some complete classification theorems. The other way is presented below and uses oligomorphocity of ω-categorical structures.

Example 12. *Consider a ternary relation R defined by $R(x_1,x_2,x_3) := (x_1 \geq x_2 \vee x_1 \geq x_3) \wedge x_1 \leq x_2 \wedge x_1 \leq x_3$ definable over $\langle \mathbb{Q}, \leq\rangle$. It can be represented by the set containing:*

- $\langle 0,0,0\rangle$*, which stands for all triples of equal rational numbers;*
- $\langle 0,0,1\rangle$*, which stands for triples of rational numbers $\langle q_1,q_2,q_3\rangle$ such that $q_1 = q_2 < q_3$;*
- *and* $\langle 0,1,0\rangle$*, which stands for triples of rational numbers $\langle q_1,q_2,q_3\rangle$ such that $q_1 = q_3 < q_2$.*

Each of the three tuples: $\langle 0,0,0\rangle, \langle 0,0,1\rangle, \langle 0,1,0\rangle$ represents some orbit of the oligomorphic structure $\langle R, \mathbb{Q}\rangle$.

Another good property of ω-categorical templates inherited from csps over finite domains is the existence of preservation theorems — see Section 2.3.

Games and qc-definitions As we noted in the previous section, sometimes it is useful to see a qc-formula ψ as a two-player game. Now we will define the notion of this game more formally. Our definitions follow those from [18].

The instance of a quantified constraint satisfaction problem is a quantified conjunctive formula ψ without free variables. The set of variables of ψ is denoted by $Var(\psi)$. The formula ψ consists of a quantifier prefix Q and a quantifier-free part ϕ. Quantifier prefix is a sequence of quantified variables of the form $Q_1x_1\ldots Q_nx_n$ where $n = |Var(\phi)|$. Each variable is either existentially or universally quantified. If a variable is existentially (universally) quantified then it is called an *existential* (*universal*) variable. The set of existential variables is denoted by $EVar(\psi)$ and the set of universal variables by $UVar(\psi)$. We say that a variable x is earlier (later) than y if it occurs earlier (later) than y in the quantifier prefix.

In the two-player game we consider, the universal and the existential player are evaluating variables with respect to the order prescribed by the quantifier prefix. The existential player wants to satisfy ϕ whereas the universal player's purpose is to falsify it. We claim that a quantified formula ψ is true if and only if the existential player can win independently of the way the universal player plays. In other words, the formula is true iff the existential player has a *winning* strategy.

Definition 13. *An existential strategy for a quantified conjunctive formula* ψ *is a sequence of mappings:*

$$\sigma = \{\sigma_i : D^{u_i} \to D\}_{x_i \in EVAR(\psi)} \tag{2.2}$$

where u_i *denotes the number of universal variables that are earlier than the existential variable* x_i.

Note that σ_i is a *Skolem function* [26] for a formula

$$Q_{i+1}x_{i+1} \dots Q_n x_n \ \phi(v_1, \dots, v_i, x_{i+1}, \dots, x_n) \tag{2.3}$$

where $v_j = x_j$ if x_j is universal and $v_j = \sigma_j(y_1, \dots, y_{u_j})$ if x_j is existential and $y_1, \dots, y_{u_j}$ are universal variables earlier than x_j.

Definition 14. *A sequence of universal moves for* ψ *is a mapping* $\delta : UVAR(\psi) \to D$.

Note that $\langle \sigma, \delta \rangle$ defined below is a valuation for variables of ψ.

Definition 15. *Let* σ *be an existential strategy and* δ *a sequence of universal moves. We put*

$$\langle \sigma, \delta \rangle(x_i) = \delta(x_i) \tag{2.4}$$

for $x_i \in UVar(\psi)$ *and*

$$\langle \sigma, \delta \rangle(x_i) = \sigma_i(\delta(x_1), \dots, \delta(x_{u_i})) \tag{2.5}$$

for $x_i \in EVar(\psi)$.

An existential strategy σ is called *winning* if for all sequences of universal moves δ the assignment $\langle \sigma, \delta \rangle : Var(\psi) \to D$ satisfies ϕ.

The following propositions are a natural result of the reasoning presented in this paragraph. Moreover, they are probably known to the reader and therefore do not require a proof.

Proposition 16. *A quantified conjunctive formula* ψ *is true if and only if there exists an existential winning strategy.*

We say that an existential player has a winning strategy if there exists an existential winning strategy. In a similar way, e.g. by negating a formula and defining dual notions to those above, we can define a *universal winning strategy*, or equivalently a winning strategy for a universal player.

Proposition 17. *A quantified conjunctive formula* ψ *is false if and only if the universal player has a winning strategy.*

2.3 Algebraic Approach

This thesis continues a line of research of classifying complexity for constraints using so-called *algebraic approach* [28, 19] to constraint satisfaction problems.

The main notion of this approach is a *polymorphism*, that is, a function that preserves a relation.

Definition 18. *Let* R *be a relation of arity* n *defined over* D. *We say that a function* $f : D^m \to D$ *is a polymorphism of* R *if for all* $a^1, \dots, a^m \in R$ *(where* a^i*, for* $1 \leq i \leq m$*, is a tuple* $\langle a^i_1, \dots, a^i_n \rangle$*), we have* $\langle f(a^1_1, \dots, a^m_1), \dots, f(a^1_n, \dots, a^m_n) \rangle \in R$. *Then we say that* f *preserves* R *or that* R *is closed under* f.

A polymorphism of Γ is a function that preserves all relations of Γ. By *Pol*(Γ) we denote the set of polymorphisms of Γ, and by *sPol*(Γ) — the set of surjective polymorphisms.

Example 19. *Consider a constraint language Γ consisting of relations: $x_1 \leq x_2$ and $x_1 < x_2$. Both of these relations are preserved by $f(x) = x$ but only the first one by $g(x) = 3$. Therefore f does and g does not belong to Pol(Γ). In this case f belongs also to sPol(Γ).*

The set of polymorphisms of an ω-categorical language Γ constitutes a *clone*, that is, a set of functions closed under composition and containing all projections. Recall that an operation π is a projection iff $\pi(x_1, \ldots, x_n) = x_i$ for all n-tuples and fixed $i \in \{1, \ldots, n\}$.

The following results link $[\Gamma]$ with *sPol*(Γ). The idea behind Theorem 20 is that the more Γ can express, in the sense of qc-definability (see Section 2.1), the less polymorphisms are contained in *sPol*(Γ). Moreover, the converse is also true. This theorem is called the *surjective preservation theorem*.

Theorem 20 ([5])**.** *Let Γ be an ω-categorical structure. Then a relation R has a qc-definition in Γ if and only if R is preserved by all surjective polymorphisms of Γ.*

Similar preservation theorems are known for (quantified) constraint satisfaction problems over finite domains [28, 12]. In the case of (not quantified) csps, there is a link between pp-definability and (not necessarily surjective) polymorphisms. However, all these algebraic tools were known before the area of csps appeared. The original proof of *Galois connection* between sets of relations and functions over finite domains was given by Bodnarčuk, Kalužnin, Kotov and Romov [42, 43]. For a recent survey on function algebras on finite sets we refer the reader to [30].

Recall that Lemma 10 links log-space reducibility and qc-definability. Combining this result and Theorem 20 we obtain the following.

Corollary 21 ([5])**.** *Let Γ_1, Γ_2 be ω-categorical structures. If sPol$(\Gamma_2) \subseteq$ sPol(Γ_1), then QCSP(Γ_1) is log-space reducible to QCSP(Γ_2).*

Let us now introduce two families of functions that are of great importance for the algebraic approach to (quantified) constraint satisfaction problems. An operation f of arity n is *essentially unary* if there exists a unary operation f_0 such that $f(x_1 \ldots, x_n) = f_0(x_i)$ for some fixed $i \in \{1, \ldots, n\}$. An operation that is not essentially unary is called *essential*. We say that a polymorphism f of an ω-categorical structure Γ is *oligopotent* if the diagonal of f, that is, the function $f(x, \ldots, x)$, is contained in the locally closed clone generated by the automorphisms of Γ. A function f of arity at least three is called a *quasi near-unanimity function* (QNUF) if it satisfies $f(x, \ldots, x, y) = \ldots = f(x, \ldots, y, x) = f(y, x, \ldots, x) = f(x, \ldots, x)$ for all $x, y \in D$.

In [7], it was shown that qcsps for ω-categorical templates closed under oligopotent quasi near-unanimity polymorphisms are polynomial. On the other hand if Γ is closed under essentially unary functions only, then we may expect that (quantified) csp is rather hard. Roughly speaking, one of the formulations of Feder-Vardi conjecture states that *CSP*(Γ) over a finite domain is NP-complete if and only if Γ is preserved by essentially unary functions only [13].

Our classification theorem in Section 1.5 gives a logical characterization for each interesting family of templates we consider. In the next chapter we formulate this theorem in terms of polymorphisms. As we will see, families of languages preserved by some surjective polymorphisms are easier than families not closed under these polymorphisms.

To give our characterization we need the following proposition and lemma, which were originally formulated and proved in Section 7 in [5]. Both of them concern equality languages (see Section 1.4). As it was stated there, each positive equality relation may be defined by a conjunction of clauses each of which is a disjunction of equalities.

Let ϕ be in conjunctive normal form and let it define some equality relation. We say that ϕ is reduced, if removing a literal or a clause from ϕ results in a formula that is not equivalent to ϕ. Clearly, every formula that defines some equality relation is equivalent to a formula in reduced form.

Proposition 22. *Let Γ be an equality constraint language over domain D. The following are equivalent.*

1. *An equality constraint language Γ is positive.*
2. *Pol(Γ) contains all essentially unary operations on D.*
3. *Pol(Γ) contains a non-injective unary operation having infinite image.*
4. *sPol(Γ) contains a non-injective unary operation.*
5. *If ϕ is a reduced formula that defines a relation from Γ, then ϕ does not contain a disequality.*
6. *$[\Gamma]$ does not contain the relation $x_1 \neq x_2$.*
7. *sPol(Γ) contains an operation that generates a constant operation.*

Lemma 23. *Let Γ be a positive equality language that is preserved by an essential operation on D with infinite image. Then Γ is preserved by all operations.*

We will also need a simple criterion of how to check whether a given formula defined using conjunction, disjunction and equality defines a positive non-negative relation. Recall from Section 1.4 that negative equality relations are those relations that can be defined by a conjunction of clauses each of which is either an equality or a disjunction of disequalities.

Lemma 24. *Let ϕ be a reduced formula consisting of disjunction, conjunction and equality only. If ϕ is not a conjunction of equalities, then it defines a positive non-negative relation.*

Proof. Consider any bijection $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$. By Lemma 23, it is enough to show that ϕ is not closed under f. Let $(x_1 = y_1 \vee \ldots \vee x_k = y_k)$ be some conjunct of ϕ such that $k > 1$. Because ϕ is in the reduced form, there is some satisfying valuation q_1 of variables in ϕ such that $q_1(x_1) = q_1(y_1)$ but $q_1(x_i) \neq q_1(y_i)$ for $2 \leq i \leq k$. Likewise, there is a satisfying valuation q_2 of variables in ϕ such that $q_2(x_2) = q_2(y_2)$ but $q_1(x_i) \neq q_1(y_i)$ for all other disjuncts. Note that since f is injective, we have that $f(q_1(x_1), q_2(x_1)) \neq f(q_1(y_1), q_2(y_1)), \ldots, f(q_1(x_k), q_2(x_k)) \neq f(q_1(y_k), q_2(y_k))$ and hence f does not preserve ϕ. □

By item 5 of Proposition 22 we have that all positive equality languages that are also negative may be defined as conjunctions of equalities. As it is not hard to show, such languages are closed under all operations on D. On the other hand, if a positive language is not negative, then similarly as in the proof of Lemma 24, we can show that some of its relations does not preserve binary injective functions. Thus, by Lemma 23 we obtain the following.

Corollary 25. *If a positive equality constraint language* Γ *over* D *is positive, but not negative, then sPol*(Γ) *is the set of all essentially unary polymorphisms on* D.

Chapter 3

Surjective Polymorphisms of Positive Temporal Relations

3.1 The Main Contribution in Terms of Polymorphisms

In this chapter we differentiate positive temporal languages with respect to their surjective polymorphisms. As we see, there are only nine maximal classes of positive temporal languages with different surjective polymorphisms. All of them are listed by Theorem 29. To show that classes α and β are different, we always give an example of a polymorphism that belongs to *sPol*(α) and does not belong to *sPol*(β) if $\beta \subseteq \alpha$; and examples of two appropriate polymorphisms if α and β are incomparable with respect to inclusion. We will now look at the surjective polymorphisms listed by Theorem 29. First, recall the definitions of surjective oligopotent QNU polymorphism and essentially unary polymorphisms from Section 2.3. As it was announced there, a qcsp for a constraint language preserved by a surjective oligopotent QNU polymorphism belongs to P. In the case of positive temporal languages, oligopotent QNU polymorphisms guarantee membership in the class of problems solvable in nondeterministic logarithmic space (NLOGSPACE). There is only one easier (in fact, not harder) case here. If a language is closed under all rational surjections, then its qcsp problem is in LOGSPACE.

Polymorphisms *spp* and *dual-spp* are surjective variants of *pp* and *dual-pp* operations introduced in [11]. As we will see these polymorphisms ensure tractability, that is, a quantified constraint satisfaction problem for a positive temporal language Γ is polynomial iff Γ is closed under *spp* or *dual-spp*.

Definition 26. *Recall that all countable dense linear orders without endpoints are isomorphic. In particular,* $\mathbb{Q}, \mathbb{Q}_-, \mathbb{Q}_+$ *and* $\mathbb{Q}_- \cup \mathbb{Q}_+$ *are isomorphic, where* $\mathbb{Q}_- = \{q \in \mathbb{Q} \mid q < 0\}$ *and* $\mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q > 0\}$. *Let* $f_1 : \mathbb{Q} \to \mathbb{Q}_-$, $f_2 : \mathbb{Q} \to \mathbb{Q}_+$ *and* $f : \mathbb{Q}_- \cup \mathbb{Q}_+ \to \mathbb{Q}$ *be any order-preserving bijections. Let*

$$spp'(a,b) = \begin{cases} a & \text{if } a < 0 \\ f_2(b) & \text{if } a \geq 0 \end{cases} \qquad dual\text{-}spp'(a,b) = \begin{cases} f_1(b) & \text{if } a \leq 0 \\ a & \text{if } a > 0 \end{cases}$$

and define $spp(a,b) = f(spp'(a,b))$ *and* $dual\text{-}spp(a,b) = f(dual\text{-}spp'(a,b))$.

Observe that *spp* is a binary surjection on $\mathbb{Q}$ such that $spp(a,b)$ is a (strict) lower bound of $\{spp(a_1,b_1),\ldots,spp(a_k,b_k)\}$ iff a is a (strict) lower bound of $\{a_1,\ldots,a_k\}$

or b is a (strict) lower bound of $\{b_1, \ldots, b_k\}$. Similarly, *dual-spp* is a surjection such that *dual-spp*(a, b) is a (strict) upper bound of $\{$*dual-spp*$(a_1, b_1), \ldots,$ *dual-spp*$(a_k, b_k)\}$ iff a is a (strict) upper bound of the set $\{a_1, \ldots, a_k\}$ or b is a (strict) upper bound of the set $\{b_1, \ldots, b_k\}$.

It appears that a logical characterization of languages closed under *spp* or *dual-spp* is uniform, too. For each $i \geq 2$ let R^i_{Left} be a positive temporal relation defined by the formula $(x_1 \leq x_2 \vee \ldots \vee x_1 \leq x_i)$. Let Γ_{Left} be a positive temporal language containing R^i_{Left} for each i. Similarly, Γ_{Right} is a set of relations defined by formulas $(x_2 \leq x_1 \vee \ldots \vee x_i \leq x_1)$. Each such formula is denoted by R^i_{Right}. As we will see, a constraint language Γ is closed under *dual-spp* iff $[\Gamma] \subseteq [\Gamma_{Left}]$. Likewise, a template Γ is closed under *spp* iff $[\Gamma] = [\Gamma_{Right}]$.

The hardest positive temporal languages are closed under some non-trivial subsets of all essentially unary polymorphisms. Unary surjections that really matter here are either *weakly monotone* or *weakly half-monotone*. As usual, we say that a function $f : \mathbb{Q} \to \mathbb{Q}$ is *weakly increasing* (*weakly decreasing*) if for all $q_1 < q_2$ we have $f(q_1) \leq f(q_2)$ (respectively $f(q_1) \geq f(q_2)$). If a function is either weakly increasing or weakly decreasing, then it is weakly monotone. Observe that the unary operation $- : \mathbb{Q} \to \mathbb{Q}$ defined as $-(x) := -x$ in usual sense is a weakly decreasing function

Definition 27. *We say that a surjection $f : \mathbb{Q} \to \mathbb{Q}$ is weakly half-increasing (respectively weakly half-decreasing) if there exist two irrational real numbers x and y such that*

- *f restricted to the set $\{q \in \mathbb{Q} \mid q < x\}$ as well as f restricted to the set $\{q \in \mathbb{Q} \mid q > x\}$ is weakly increasing (respectively, weakly decreasing), and*
- *for all $q < x$ we have $f(q) > y$ (respectively $f(q) < y$) and for all $q > x$ we have $f(q) < y$ (respectively $f(q) > y$).*

A function is weakly half-monotone if it is weakly half-increasing or weakly half-decreasing.

Example 28. *Recall that all countable dense linear orders without endpoints are isomorphic. In particular, $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are isomorphic, so we may identify $\mathbb{Q}$ with $\mathbb{Q} \setminus \{0\}$ and think of 0 as an irrational number in $\mathbb{Q} \setminus \{0\}$. Then the function $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ defined by $f(q) = \frac{1}{q}$ is weakly half-decreasing and the function defined by $f(q) = \frac{-1}{q}$ is weakly half-increasing.*

The simplest relation closed under a half-increasing function is a 3-ary 'cyclic' relation: $(x_1 \leq x_2 \leq x_3) \vee (x_2 \leq x_3 \leq x_1) \vee (x_3 \leq x_1 \leq x_2)$. The exact correspondence between half-increasing functions and 'cyclic' positive temporal relations is given in Section 3.6.

Other representatives closed under unary functions only are given in the theorem below.

Now, we give a formulation of Theorem 6 in terms of polymorphisms. Item (i) for $1 \leq i \leq 5$ from Theorem 6 refers to the item of the same number from the following theorem. However item 6 must be splitted into four cases if we speak in the language of functions. Recall from Section 2.1 that for all constraint languages Γ_1, Γ_2 we have $[\Gamma_1] = [\Gamma_2]$ if and only if *sPol*$(\Gamma_1) =$ *sPol*(Γ_2).

To shorten the presentation we formulate Theorem 29 below as one theorem of the form: *exactly one of the following holds: C_1 and then ϕ_1; $\ldots$ C_9 and then ϕ_9*. In fact this theorem should be understood as a collection of 10 theorems: one saying that exactly one of the conditions $C_1 \dots C_9$ is true, and then 9 theorems saying: *if C_i is true, then ϕ_i is true*.

Theorem 29. *Let Γ be a positive temporal language, then exactly one of the following holds.*

1. *The set sPol(Γ) is equal to the set of all rational surjections; and then $[\Gamma]$ is equal to $[x_1 = x_2]$. Each relation in Γ is definable as a conjunction of equalities.*

2. *The set sPol(Γ) contains a surjective oligopotent QNU polymorphism, but it does not contain the function $-$; and then $[\Gamma]$ is equal to $[x_1 \leq x_2]$. Each relation in Γ is definable as a conjunction of inequalities.*

3. *The set sPol(Γ) contains neither a surjective oligopotent QNU polymorphism nor a dual-spp polymorphism, but it contains spp; and then $[\Gamma]$ is equal to $[\Gamma_{Right}]$. Each relation in Γ is definable by the formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$.*

4. *The set sPol(Γ) contains neither a surjective oligopotent QNU polymorphism nor a spp polymorphism, but it contains dual-spp; and then $[\Gamma]$ is equal to $[\Gamma_{Left}]$. Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$.*

5. *The set sPol(Γ) is the set of all essentially unary surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[x_1 = x_2 \vee x_1 = x_3]$. Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} = y_{i_1} \vee \ldots \vee x_{i_k} = y_{i_k})$.*

6. *The set sPol(Γ) is the set of all weakly increasing, weakly decreasing or weakly half-monotone surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[R]$ such that the definition of R is a conjunction of clauses $(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)} \vee x_{\Pi(3)} \leq x_{\Pi(4)})$ where Π ranges over all cycles and reversed cycles of the set $\{1, 2, 3, 4\}$.*

7. *The set sPol(Γ) is the set of all weakly increasing or weakly decreasing surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_3 \leq x_2 \vee x_2 \leq x_1)]$.*

8. *The set sPol(Γ) is the set of all weakly increasing or weakly half-increasing surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_2 \leq x_3 \vee x_3 \leq x_1) \wedge (x_3 \leq x_1 \vee x_1 \leq x_2)]$.*

9. *The set sPol(Γ) is the set of all weakly increasing surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[(x_1 \leq x_2 \vee x_2 \leq x_3)]$.*

Outline of the chapter. As in it is not hard to see, each temporal relation can have many different logical representations. In this thesis we focus on one of them, which is closely related to so-called *filter representation* of temporal relations and will be presented in Section 3.2. We introduced positive temporal relations as those that have a positive definition over $\langle Q, \leq \rangle$. In Section 3.3, we show that a temporal relation is positive iff it is closed under all weakly increasing functions. We give there also a characterization of temporal languages in terms of filters. In our opinion, all these properties confirm that positive temporal relations form a natural subset of all temporal relations. The whole chapter 3 is devoted to the proof of Theorem 29. In sections 3.4 and 3.5, we derive first four cases of this theorem, that is, we show that a positive temporal language belongs to one of the first four families or it is closed under essentially unary polymorphisms only. Section 3.6 presents the proof of the fact that there are only five different maximal sets of positive temporal languages preserved by unary surjections only.

3.2 Filter Representation of Temporal Relations

Here, after a few definitions we give some representation of temporal languages. At first look it may look somewhat confusing, but Example 31 should clarify our point.

A preorder is a reflexive and transitive relation. A preorder $\preceq$ on a set A is total if for all $a, b \in A$ we have $a \preceq b$ or $b \preceq a$. We call A the domain of $\preceq$ and we write $A = Dom(\preceq)$. We use $a \prec b$ as an abbreviation for $a \preceq b \wedge b \not\preceq a$ and $a \approx b$ as an abbreviation for $a \preceq b \wedge b \preceq a$. In the following we represent total preorders on finite sets of variables as sequences of the form $x_1 \sim_1 x_2 \sim_2 \ldots \sim_{n-1} x_n$ where each $\sim_i$ is either $\prec$ or $\approx$ and $\{x_1, \ldots, x_n\} = Dom(\preceq)$. For example $a \prec b \approx c$ is the representation of $\preceq = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle, \langle c, b\rangle\}$. By *Range*$(\preceq)$ we denote the size of a maximal set of variables $\{x_{i_1}, \ldots, x_{i_l}\} \subseteq Dom(\preceq)$ such that for all pairs x_{i_p}, x_{i_r} we have either $x_{i_p} \prec x_{i_r}$ or $x_{i_r} \prec x_{i_p}$.

We write $\preceq_1 \lhd \preceq_2$ and say that $\preceq_1$ is more general than $\preceq_2$ if $\preceq_1$ is a restriction of $\preceq_2$ to a smaller domain. Formally, $\preceq_1 \lhd \preceq_2$ if $Dom(\preceq_1) \subseteq Dom(\preceq_2)$ and $\preceq_1 = \preceq_2 \cap (Dom(\preceq_1) \times Dom(\preceq_1))$. We write $\preceq_1 \ll \preceq_2$ and say that $\preceq_1$ is flatter than $\preceq_2$ if $Dom(\preceq_1) = Dom(\preceq_2)$ and $\preceq_2$ as a relation is a subset of $\preceq_1$ (see the example below).

We say that a valuation $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ is compatible with a total preorder $\preceq$ on $\{x_1, \ldots, x_n\}$ if for all x_i, x_j such that $x_i \preceq x_j$ we have $q(x_i) \leq q(x_j)$. We then also say that the tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ is compatible with $\preceq$.

Definition 30. *Consider a temporal relation $R(x_1, \ldots, x_n)$. We say that $\preceq$ is a* bound *for R if $\preceq$ is a total preorder on a subset of $\{x_1, \ldots, x_n\}$ such that for all valuations $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ compatible with $\preceq$ the tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ is* not *in R. The set of bounds of R is denoted $\mathcal{B}(R)$. A minimal wrt $\lhd$ bound of R is called a* filter *for R. The set of filters of R is denoted $\mathcal{F}(R)$.*

Let Γ be a temporal language. Assume that each $R \in \Gamma$ is defined over different set of variables. Then by $\mathcal{F}(\Gamma)$ equal to $\bigcup_{R \in \Gamma} \mathcal{F}(R)$ we denote a set of filters of Γ. Similarly, we write $\mathcal{B}(\Gamma)$ for a set of bounds of Γ.

Example 31. *Let R be a temporal relation defined by $(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_2 \leq x_1)$. Then $x_3 \prec x_2 \prec x_1$, $x_1 \prec x_2$ and $x_1 \approx x_3 \prec x_2$ are bounds for R (in fact R has more bounds). The bound $x_1 \approx x_3 \prec x_2$ is not a filter because $x_1 \prec x_2$ is more general. The relation R' defined by $(x_1 \leq x_2 \vee x_1 \leq x_3)$ has three filters: $x_3 \prec x_2 \prec x_1$, $x_3 \approx x_2 \prec x_1$ and $x_2 \prec x_3 \prec x_1$. The filter $x_3 \approx x_2 \prec x_1$ is flatter than both $x_3 \prec x_2 \prec x_1$ and $x_2 \prec x_3 \prec x_1$. The range of a preorder $x_3 \approx x_2 \prec x_1$ is 2, but the range of a preorder $x_3 \prec x_2 \prec x_1$ is equal to 3.*

In this thesis we represent temporal relations (languages) with their sets of filters. Consider the situation where we have some $\mathcal{F}(\Gamma)$ and we ask for $\mathcal{F}(R)$ of some relation R that is qc-definable over Γ; and the converse situation: when we want to infer something about $\mathcal{F}(\Gamma)$ from $\mathcal{F}(R)$. The following lemmas give us a partial answer for such questions. In lemmas 32–35 the relation R_1 belongs to $[R]$. These lemmas are used in each of the following sections of the chapter and therefore are of crucial importance.

Let $\preceq_{Var}$ be a preorder such that $x \approx y$ for all $x, y \in Var$.

Lemma 32. *Let $R(x_1, \ldots, x_n)$ be a temporal relation. Consider $R_1(x_k, \ldots, x_n)$ defined by $\exists x_1 \ldots \exists x_{k-1}\ R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{k-1} x_i = x_{i+1}$. If a preorder $\preceq_1$ is a bound of R_1, then each preorder $\preceq$ such that $\preceq_1 \lhd \preceq$ and $x_i \approx x_{i+1}$ for $1 \leq i \leq k-1$ is a bound of R.*

Proof. Assume on the contrary that some $\preceq$, as defined above, does not belong to $\mathcal{B}(R)$. Therefore there exists $\langle q(x_1), \ldots, q(x_n)\rangle \in R$ for some $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ that is compatible with $\preceq$. Then q is compatible with $\preceq_1$ and with $\preceq_{\{x_1,\ldots,x_k\}}$. In particular $q(x_1) = \ldots = q(x_k)$. Therefore $t = \langle q(x_k), \ldots, q(x_n)\rangle$ is in R_1. Then t is compatible with $\preceq_1$ and hence $\preceq_1$ cannot be a bound of R_1. □

Lemma 33. *Let $R(x_1, \ldots, x_n)$ be a temporal relation with a filter $\preceq$ such that $x_i \approx x_{i+1}$ for $i = 1, \ldots, k-1$. Consider a relation $R_1(x_k, \ldots, x_n)$ defined by $\exists x_1 \ldots \exists x_{k-1}$ $R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{k-1} x_i = x_{i+1}$. Then either a restriction of $\preceq$ to $\{x_k, \ldots, x_n\}$ or a restriction of $\preceq$ to $\{x_{k+1}, \ldots, x_n\}$ is a filter of R.*

Proof. Let a preorder $\preceq_1$ be a restriction of $\preceq$ to $\{x_k, \ldots, x_n\}$. First, we show that $\preceq_1$ is a bound of R_1. Assume the contrary. Then there exists a tuple $\langle q(x_k), \ldots, q(x_n)\rangle \in R_1$ compatible with $\preceq_1$. By the definition of the relation R_1, there exists a tuple $t = \langle q_1(x_1), \ldots, q_1(x_k), \ldots, q_1(x_n)\rangle$ in R such that $q_1(x_1) = \ldots = q_1(x_k)$ and $q_1(x_i) = q(x_i)$ for $k \leq i \leq n$. Observe that t is compatible with $\preceq$ and hence $\preceq$ is not a bound of R.

Now, we show that one of these preorders is a filter of R_1. Assume on the contrary that neither of them is a filter of R_1. Then there exists a filter $\preceq_1'$ such that $x_l \notin Dom(\preceq_1')$ for some $l > k$. Now, we can use Lemma 32 to show that there exists a bound of R that is more general than $\preceq$. In fact, in this way we obtain a restriction of $\preceq$ to some smaller domain not containing x_l. □

Example 34. *Consider the preceding lemma once more. At first glance it may seem that a restriction of $\preceq$ to $\{x_k, \ldots, x_n\}$ is always a filter of R_1 and the second case is unnecessary there. To see that it is not the case, consider the following relation.*

Let $R(x_1, x_2, x_3, x_4)$ be defined by $(x_1 = x_2 \vee x_1 = x_3 \vee x_1 = x_4) \wedge (x_2 = x_1 \vee x_2 = x_3 \vee x_2 = x_4) \wedge (x_3 = x_1 \vee x_3 = x_2 \vee x_3 = x_4) \wedge (x_4 = x_1 \vee x_4 = x_2 \vee x_4 = x_3)$. Then $x_1 \approx x_2 \approx x_3 \prec x_4$ is a filter of R. Moreover, let a relation $R_1(x_2, x_3, x_4)$ be equivalent to $R(x_1, x_2, x_3, x_4) \wedge x_1 = x_2$. Now, the reader can convince himself that $x_3 \prec x_4$ and not $x_2 \approx x_3 \prec x_4$ is a filter of R_1.

Lemma 35. *Consider $R(x_1, \ldots, x_n)$ and $R_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ defined by $\exists x_i\ R(x_1, \ldots, x_n)$. Then a preorder $\preceq$ such that $x_i \notin Dom(\preceq)$ is a filter of R if and only if it is a filter of R_1.*

Proof. For the sake of simplicity we assume that $i = 1$. First we show that $\preceq$ is a bound of R if and only if it is a bound of R_1. Then 'we turn to filters'.

(Part One) Assume on the contrary that $\preceq \in \mathcal{B}(R)$ and $\preceq \notin \mathcal{B}(R_1)$. Then there exists a tuple $\langle q_1(x_2), \ldots, q_1(x_n)\rangle \in R_1$ compatible with $\preceq$ where $q_1 : \{x_2, \ldots, x_n\} \to \mathbb{Q}$. By the definition of R_1, there exists some $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ such that $t = \langle q(x_1), \ldots, q(x_n)\rangle$ belongs to R and $q(x_i) = q_1(x_i)$ for all $2 \leq i \leq n$. Note that t is compatible with $\preceq$ and therefore $\preceq$ cannot be a bound of R.

(Part Two) Assume on the contrary that $\preceq \in \mathcal{B}(R_1)$ and $\preceq \notin \mathcal{B}(R)$. Then for some $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ there exists a tuple $\langle q(x_1), \ldots, q(x_n)\rangle \in R$ compatible with

$\preceq$. By the definition of R_1, a tuple $t_1 = \langle q(x_2), \ldots, q(x_n)\rangle$ belongs to R_1. Since t_1 is compatible with $\preceq$, this preorder is not a bound of R_1.

(Part Three) Here we show that if $\preceq \in \mathcal{F}(R)$, then $\preceq \in \mathcal{F}(R_1)$. Assume on the contrary that there exists some $\preceq' \in \mathcal{B}(R_1)$ such that $\preceq' \triangleleft \preceq$. Then, by (Part Two), the preorder $\preceq'$ is a bound of R. Because $\preceq' \triangleleft \preceq$, we reach a contradiction.

(Part Four) What remains to be shown is: if $\preceq \in \mathcal{F}(R_1)$, then $\preceq \in \mathcal{F}(R)$. Again, assume on the contrary that there exists some $\preceq' \in \mathcal{B}(R)$ such that $\preceq' \triangleleft \preceq$. Here we use (Part One) to show that it contradicts with: $\preceq$ being a filter of R_1. □

3.3 Positive Temporal Relations

An arbitrary temporal relation is closed under all unary strictly increasing functions — they are automorphisms of such a relation. As we will see, a temporal relation is positive iff it is closed under all weakly increasing unary rational functions. First we will show that each temporal relation is closed under all weakly increasing surjections. To prove the converse we will need a characterization od positive temporal languages in terms of filters.

Lemma 36. *Let $R(x_1, \ldots, x_n)$ be a positive temporal relation. Then it is closed under all weakly increasing functions $f : \mathbb{Q} \to \mathbb{Q}$.*

Proof. Assume that a definition $\phi(x_1, \ldots x_n)$ of R is in the CNF form. Let a tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ be in R. Take any clause $(x_{i_1} \leq x_{j_1} \vee \ldots \vee x_{i_k} \leq x_{j_k})$. Now, note that if $q(x_{i_p}) \leq q(x_{j_p})$, for some $1 \leq p \leq k$, then also $f(q(x_{i_p})) \leq f(q(x_{j_p}))$. Therefore each clause of ϕ is satisfied by $\langle f(q(x_1)), \ldots, f(q(x_n))\rangle$ and hence R is closed under f. □

We said that a preorder $\preceq$ is a bound for R if none of tuples compatible with this preorder belongs to R. In fact, if one of such tuples does not belong to R, then none of them does. It is what the next lemma states.

Lemma 37. *Let $R(x_1, \ldots, x_n)$ be a positive temporal relation and $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ be a valuation of variables of R in Q such that $\langle q(x_1), \ldots, q(x_n)\rangle \notin R$. Then each preorder $\preceq$ compatible with q is a bound of R.*

Proof. Assume on the contrary that a preorder $\preceq$ compatible with q is not a bound of R. Then there exists some tuple $\langle q_1(x_1), \ldots, q_1(x_n)\rangle \in R$ compatible with $\preceq$ where $q_1 : \{x_1, \ldots, x_n\} \to \mathbb{Q}$. Because both q and q_1 are compatible with $\preceq$, there exists some strictly increasing unary function f such that $\langle f(q_1(x_1)), \ldots, f(q_1(x_n))\rangle$ is equal to $\langle q(x_1), \ldots, q(x_n)\rangle$. Hence by Lemma 36, we have that R is preserved by f and $\langle q(x_1), \ldots, q(x_n)\rangle \in R$, which contradicts our assumption. □

It is rather obvious that positive temporal relations have substantially less varied structure than arbitrary temporal relations. The dependencies between tuples in positive temporal relations can be given in terms of the order $\ll$. The following results unfold the reason why we introduced this order. Recall: if $\preceq \ll \preceq_1$, then we say that $\preceq$ is flatter than $\preceq_1$.

Lemma 38. *Let a preorder $\preceq$ be flatter than $\preceq_1$ and $Dom(\preceq) \subseteq \{x_1, \ldots, x_n\}$. Consider a tuple $\langle q(x_1), \ldots, q(x_n)\rangle$ compatible with $\preceq_1$ where $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$. Then there exists a weakly increasing function f such that $\langle f(q(x_1)), \ldots, f(q(x_n))\rangle$ is compatible with $\preceq$.*

Proof. Let $\preceq_b^1 \ll \ldots \ll \preceq_b^k$ be the longest sequence of pairwise different preorders such that $\preceq_b^1$ is equivalent to$\preceq$ and $\preceq_b^k$ to $\preceq_1$. We assume that there is a tuple $t = \langle q(x_1), \ldots, q(x_n)\rangle$ compatible with the preorder $\preceq_b^{i+1}$ and then claim that there exists a function f_i such that $\langle f_i(q(x_1)), \ldots, f_i(q(x_n))\rangle$ is compatible with $\preceq_b^i$. Finally, we obtain f as a composition of functions $f_{n-1} \cdot \ldots \cdot f_1$.

Note that for $\preceq_b^i, \preceq_b^{i+1}$ there are exactly two sets of variables $\{x_{j_1}, \ldots, x_{j_p}\}$ and $\{x_{k_1}, \ldots, x_{k_r}\}$ such that $x_{j_1} \approx_b^i \ldots \approx_b^i x_{j_p} \approx_b^i x_{k_1} \approx_b^i \ldots \approx_b^i x_{k_r}$, but $x_{j_m} \prec_b^{i+1} x_{k_n}$ for all $1 \leq m \leq p$ and $1 \leq n \leq r$.

Define $f_i : \mathbb{Q} \to \mathbb{Q}$ to be constant on the interval $[q(x_{j_1}), q(x_{k_1})]$ and strictly increasing everywhere else. Since t is compatible with $\preceq_b^{i+1}$, there is no $q(x_k)$ that lies in the interval $(q(x_i), q(x_j))$ and therefore $\langle f_i(q(x_1)), \ldots, f_i(q(x_n))\rangle$ is compatible with $\preceq_b^i$. □

Lemma 39. *Let $R(x_1, \ldots, x_n)$ be a temporal relation. Then it is a positive temporal relation if and only if for all bounds $\preceq$ of R each preorder $\preceq_1$ such that $\preceq \ll \preceq_1$ is also a bound of R.*

Proof. (Part One) We begin with the left-to-right implication. Assume on the contrary that there are preorders $\preceq \ll \preceq_1$ such that $\preceq \in \mathcal{B}(R)$ and $\preceq_1 \notin \mathcal{B}(R)$. Then there exists a tuple $\langle q(x_1), \ldots, q(x_n)\rangle \in R$ compatible with $\preceq_1$. By Lemma 38, there is some weakly increasing function f such that $\langle f(q(x_1)), \ldots, f(q(x_n))\rangle$ is compatible with $\preceq$. Since, by Lemma 36, the relation R is closed under f, we have a contradiction.

(Part Two) Now, we show the right-to-left implication. Let R be a temporal relation such that for all its bounds $\preceq$ each preorder $\preceq_1$ satisfying $\preceq \ll \preceq_1$ is also a bound of R. We now give a definition ϕ_R of R using only $\vee, \wedge$, and $\leq$. For each minimal with respect to $\ll$ bound $x_1^1 \approx \ldots \approx x_{l_1}^1 \prec \ldots \prec x_1^n \approx \ldots \approx x_{l_n}^n$ of R, the definition ϕ_R contains a clause $C_{\preceq}$ of the form $(\bigvee_{i=1}^{n-1} \bigvee_{j=i+1}^{n} \bigvee_{k=1}^{l_i} \bigvee_{m=1}^{l_j} x_k^i \geq x_m^j)$.

To complete the proof we show that a tuple $\vec{t} := \langle q(x_1), \ldots, q(x_n)\rangle$ for some $q : \{x_1, \ldots, x_n\} \to Q$ does not belong to R if and only if q does not satisfy ϕ_R. If $\vec{t} \notin R$, then, by Lemma 37, there is some bound $\preceq$ of R compatible with $\vec{t}$. Let a preorder $\preceq'$ be flatter than $\preceq$ and minimal with respect to $\ll$ in $\mathcal{B}(R)$. Then a formula ϕ_R contains a clause $C_{\preceq'}$ of the form $(\bigvee_{i=1}^{n-1} \bigvee_{j=i+1}^{n} \bigvee_{k=1}^{l_i} \bigvee_{m=1}^{l_j} x_k^i \geq x_m^j)$, which is satisfied if there exists $1 \leq i < j \leq n$ such that a value of x_k^i is greater than or equal to the value assigned to x_m^j for some $1 \leq k \leq l_i$ and $1 \leq m \leq l_j$. Therefore, all tuples compatible with some preorder less flat than $\preceq'$ do not satisfy $C_{\preceq'}$. Conversely, if a valuation q does not satisfy ϕ_R, then there is some clause $C_{\preceq}$ that is falsified by q. As it was explained, the index of $C_{\preceq}$ corresponds to some minimal with respect to $\ll$ bound of R. Let $x_1, \ldots, x_n$ be variables occurring in $C_{\preceq}$. Note that if $\langle q(x_1), \ldots, q(x_n)\rangle$ does not satisfy $C_{\preceq}$, then it is compatible with $\preceq$. □

Note that, by Lemma 39, bounds less flat than or equal to $x_1 \approx \ldots \approx x_n$ forbid all possible valuations of $x_1, \ldots, x_n$ in $\mathbb{Q}$. Therefore we have the following.

Corollary 40. *Let $R(x_1, \ldots, x_n)$ be a non-empty positive temporal relation. For all nonempty subsets Var of $\{x_1, \ldots, x_n\}$ the preorder $\preceq_{Var}$ is not a bound of R.*

Note that if we have two bounds comparable wrt $\ll$ then by Lemma 39 the flatter of them is more interesting. Therefore we focus on minimal wrt $\ll$ filters. We illustrate this lemma on the following example.

Example 41. *Consider once more a positive temporal relation given by $(x_1 \leq x_2 \vee x_1 \leq x_3)$. It has a filter $x_2 \approx x_3 \prec x_1$, but also filters (bounds) $x_2 \prec x_3 \prec x_1$*

and $x_3 \prec x_2 \prec x_1$. Note or recall from Example 31 that the first of the mentioned preorders is flatter than the remaining two. Moreover, these are all filters of the relation $(x_1 \leq x_2 \vee x_1 \leq x_3)$.

Finally, we can give a characterization of positive temporal templates in terms of polymorphisms.

Lemma 42. *A temporal relation $R(x_1, \ldots, x_n)$ is positive if and only if it is closed under all weakly increasing functions $f : \mathbb{Q} \to \mathbb{Q}$.*

Proof. The left-to-right implication follows from Lemma 36. To prove the right-to-left implication, we will use Lemma 39. We will show that if $R(x_1, \ldots, x_n)$ is closed under all weakly increasing functions, then provided $\preceq$ is a bound of R, each preorder $\preceq_1$ less flat than $\preceq$ is a bound of R. Assume on the contrary that some $\preceq$ is a bound of R but $\preceq_1$ such that $\preceq \ll \preceq_1$ is not. Then for some $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$ there exists a tuple $\langle q(x_1), \ldots, q(x_n) \rangle \in R$ compatible with $\preceq_1$. However, using Lemma 38 there exists a unary weekly increasing function f such that $\langle f(q(x_1)), \ldots, f(q(x_n)) \rangle$ is compatible with $\preceq$. Since f is a polymorphism of R, we have a contradiction. □

We finish this section by one more auxiliary lemma.

Lemma 43. *Let $R(x_1, \ldots, x_n)$ be a temporal relation. Let $\preceq$ be a minimal with respect to $\ll$ filter of R such that $x_i \approx x_{i+1}$ for $i = 1, \ldots, k-1$. Let $R_1(x_k, \ldots, x_n)$ be defined as $\exists x_1 \ldots \exists x_{k-1}\ R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{k-1} x_i = x_{i+1}$. Then there exists a preorder $\preceq_1$ in $\mathcal{F}(R_1)$ such that $\preceq_1 \triangleleft \preceq$ and Range$(\preceq_1) =$ Range$(\preceq)$.*

Proof. By Lemma 33, there exists $\preceq_1 \in \mathcal{F}(R_1)$ that is either a restriction of the preorder $\preceq$ to $\{x_k, \ldots, x_n\}$ or to $\{x_{k+1}, \ldots, x_n\}$. If $x_k \in$ *Dom*$(\preceq_1)$, then it is easily seen that *Range*$(\preceq_1) =$ *Range*$(\preceq)$.

Now, we consider the second case. Assume on the contrary that *Range*$(\preceq_1) <$ *Range*$(\preceq)$. Since $x_k \notin$ *Dom*$(\preceq_1)$, we can use Lemma 32 to show that there exists $\preceq_2$ (a bound of R) satisfying $\preceq_2 \ll \preceq$. If $\preceq_2$ is a filter, then $\preceq$ is not minimal wrt. $\ll$ in $\mathcal{F}(R)$ and hence we have a contradiction. Otherwise, there must be some filter $\preceq_3$ of R more general than $\preceq_2$. By Lemma 39, each preorder $\preceq_4$ such that $\preceq_3 \ll \preceq_4$ is a bound of R. Observe that among them there is some preorder $\preceq_4$ more general than $\preceq$ and hence $\preceq$ is not a filter of R. □

3.4 Non Unary Surjective Polymorphisms of Positive Temporal Relations

In this section we derive the first four classes of Theorem 29. We show that a positive temporal language belongs to one of these four classes, or it is closed under essentially unary surjective polymorphisms only.

In particular we show that a positive temporal language is closed under a binary surjective polymorphism *spp*, under a binary surjective polymorphism *dual-spp*, or it is preserved by essentially unary surjective polymorphisms only. Recall definitions of these polymorphisms from Section 3.1.

3.4.1 Classes of different power of qc-definability

The topic of this subsection is summarized by the following theorem. Recall from Section 2.1 that $[\Gamma]$ is the set of all relations that are qc-definable by relations of Γ. Note that by Lemma 10, the problems $QCSP(\Gamma_1)$ and $QCSP(\Gamma_2)$ for some positive temporal languages Γ_1 and Γ_2 such that $[\Gamma_1] = [\Gamma_2]$ are logspace equivalent.

Theorem 44. *Let Γ be a positive temporal language, then exactly one of the following holds.*

1. *The set sPol(Γ) is equal to the set of all rational surjections; and then $[\Gamma]$ is equal to $[x_1 = x_2]$. Each relation in Γ is definable as a conjunction of equalities.*
2. *The set sPol(Γ) contains a surjective oligopotent QNU polymorphism, but it does not contain the function $-$; and then $[\Gamma]$ is equal to $[x_1 \leq x_2]$. Each relation in Γ is definable as a conjunction of inequalities.*
3. *The set sPol(Γ) contains neither a surjective oligopotent QNU polymorphism nor a dual-spp polymorphism, but it contains spp; and then $[\Gamma]$ is equal to $[\Gamma_{Right}]$. Each relation in Γ is definable by the formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$.*
4. *The set sPol(Γ) contains neither a surjective oligopotent QNU polymorphism nor a spp polymorphism, but it contains dual-spp; and then $[\Gamma]$ is equal to $[\Gamma_{Left}]$. Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$.*
5. *Each relation in Γ is preserved by essentially unary surjective polymorphisms only.*

First, we want to distinguish $[\Gamma_{Left}]$ and $[\Gamma_{Right}]$ from each other and then from $[(x_1 \leq x_2)]$. Recall that in our language, see Theorem 20, sets $[\Gamma_1]$ and $[\Gamma_2]$ for some positive temporal relations Γ_1 and Γ_2 are different if there exists a function that preserves relations from exactly one of these sets.

Lemma 45. *The language Γ_{Left} is closed under dual-spp operation but it is not closed under spp. Dually, Γ_{Right} is closed under spp but it is not closed under dual-spp.*

Proof. (Part One) First we show that each relation R^i_{Left} is closed under the function *dual-spp*. Note that a tuple $\langle q(x_1), \ldots, q(x_i)\rangle$ is not in R^i_{Left} if and only if $q(x_1)$ is a strict upper bound of the set $\{q(x_2), \ldots, q(x_i)\}$. Moreover, by the remark following Definition 26, the function *dual-spp* cannot return such a tuple operating on $\langle q_1(x_1), \ldots, q_1(x_i)\rangle$ and $\langle q_2(x_1), \ldots, q_2(x_i)\rangle$ such that neither $q_1(x_1)$ is a strict upper bound of the set $\{q_1(x_2), \ldots, q_1(x_i)\}$ nor $q_1(x_1)$ is a strict upper bound of the set $\{q_1(x_2), \ldots, q_1(x_i)\}$.

(Part Two) We now show that R^3_{Left} is not closed under *spp*. Consider tuples: $\langle 1, -1, 1\rangle$ and $\langle 0, 0, -1\rangle$. Both of them are in R^3_{Left}, but *spp*$(1, 0)$ is strictly greater than *spp*$(-1, 0)$ and *spp*$(1, -1)$. Hence R^3_{Left} and Γ_{Left} are not closed under *spp*.

(Part Three) Proofs for *spp* are dual and therefore omitted. □

It is quite obvious that $(x_1 \leq x_2)$ is closed under both *spp* and *dual-spp*. In [7], it is shown that $(x_1 \leq x_2)$ is closed under *median*, which is the ternary function that returns the median of its three argument. It is not hard to show that *median* is a surjective oligopotent QNU polymorphisms. To distinguish $(x_1 \leq x_2)$ from Γ_{Left} and Γ_{Right} we show that the last two relations are not closed under any surjective QNU polymorphism.

Lemma 46. *Neither Γ_{Left} nor Γ_{Right} is closed under any surjective oligopotent QNU polymorphism.*

(Part One) An n-ary function f is called ord^n polymorphism if $f(a_1, \dots, a_n) \le f(b_1, \dots, b_n)$ whenever $a_i \le b_i$ for each $1 \le i \le n$. Note that the relation R^2_{Left} defined by $(v_1 \le v_2)$ is closed only under ord^n polymorphisms, that is, each function from *sPol*(R^2_{Left}) is an ord^n function for some n. Therefore the template Γ_{Left} is closed only under ord^n functions. As we will show, there is no surjective function that is both the ord^n operation and an oligopotent QNU polymorphism.

(Part Two) Suppose that g is both a k-ary surjective oligopotent QNU polymorphism and an ord^k function. Let $g(a, \dots, a) = a'$ for some $a, a' \in Q$. Since g is surjective, there exist $b_1, \dots, b_k$ such that $g(b_1, \dots, b_k) > a'$. Because g is an ord^k function, we have that $g(c_1, \dots, c_k) > a'$ where each $c_i = \max(a, b_i)$. Further, it holds also that $g(d, \dots, d)$, where $d = \max(c_1, \dots, c_k)$, is strictly greater than a'. Because g is a QNU polymorphism, we have that $g(a, \dots, a)$ is equal to $g(d, a, \dots, a) = \dots = g(a, \dots, a, d)$, but it is strictly less than $g(d, \dots, d)$.

(Part Three) We use the same notation as in (Part Two). Assume on the contrary that there is some k-ary oligopotent QNU polymorphism g in *sPol*(Γ_{Left}). We show that R^{k+1}_{Left} is not closed under g. Let $\vec{t_i} = \langle q_i(x_1), \dots, q_i(x_{k+1})\rangle$ for $1 \le i \le k$ be defined as follows. We set $q_i(x_l)$ to d if $l = 1$ or $l = i + 1$, otherwise we set $q_i(x_l)$ to a. Since for each $2 \le i \le k$ there exists l such that $q_i(x_1) = q_i(x_l)$, we have that all $\vec{t_i}$ belong to R^{k+1}_{Left}. Further $s_1 = g(q_1(x_1), \dots, q_k(x_1))$ is equal to $g(d, \dots, d)$ and $s_i = g(q_1(x_i), \dots, q_k(x_i))$ to $g(a, \dots, d, a, \dots, a)$ where d is the $(i-1)$-th argument and $2 \le i \le k + 1$. Using (Part Two), we have that for all $2 \le i \le k + 1$, the value of s_1 is strictly greater than s_i. Therefore the tuple $\langle s_1, \dots, s_{k+1}\rangle$ is not in R^{k+1}_{Left}. Hence this relation is not closed under g. Because g was arbitrary we infer that Γ_{Left} is not closed under any surjective oligopotent QNU polymorphism.

(Part Four) The proof concerning Γ_{Right} is dual and omitted. □

It turns out that the relation defined by $(x_1 \le x_2 \vee x_1 \le x_3)$ has the same expressive power, in the sense of qc-definability, as Γ_{Left}. In the same context the relation defined by $(x_2 \le x_1 \vee x_3 \le x_1)$ is as powerful as the whole Γ_{Right}.

Lemma 47. *Every relation in Γ_{Left} has a qc-definition over $(x_1 \le x_2 \vee x_1 \le x_3)$, that is, $[(x_1 \le x_2 \vee x_1 \le x_3)] = [\Gamma_{Left}]$. Similarly, $[(x_2 \le x_1 \vee x_3 \le x_1)] = [\Gamma_{Right}]$.*

Proof. We show only that $[x_1 \le x_2 \vee x_1 \le x_3] = [\Gamma_{Left}]$. Note that it will be enough to prove that for each $i \ge 4$ the relation R^i_{Left} may be defined using only relations R^k_{Left} where $3 \le k < i$, then the result follows by induction on i.

Let $Z = \{x_2, \dots, x_i\}$. To show the induction statement we define a formula using variables $x_1, \dots, x_i$ but we also introduce new ones: a variable x_Z and one x_S for each subset S of Z of the size $i - 2$. All of these variables are existentially quantified and we build with them the following clauses.

- For each S we have a clause of the form $(x_S \le x_{e_1} \vee \dots \vee x_S \le x_{e_{i-2}})$ where $\{x_{e_1}, \dots, x_{e_{i-2}}\} = S$
- For each S there is a clause $(x_Z \le x_S \vee x_Z \le x_e)$ where $\{e\} = Z \setminus S$.
- There is a clause $x_1 \le x_Z$.

Obviously, the conjunction of these clauses is a qc-formula over $\{R^2_{Left}, \dots, R^{i-1}_{Left}\}$. We now show that it is indeed a definition of R^i_{Left}. We first claim that the definition does not allow x_1 to be strictly greater than all of $x_2, \dots, x_i$. Assume the contrary. If we set all $x_2, \dots, x_i$ to be strictly less than x_1, then, by the first kind of clauses, each variable x_S must be also strictly less than x_1. Further, by the second kind of clauses x_Z must be strictly less than x_1. But such a valuation falsify the last clause and hence we reach a contradiction.

It remains to prove the converse. Suppose that $\langle q(x_1), \dots, q(x_i)\rangle \in R^i_{Left}$. That is, some x_k, for $2 \le k \le i$, is set to be greater than or equal to x_1. We show a valuation of existential variables that satisfies all kinds of clauses. Set x_Z as well as each x_S to be equal to the value of the greatest element of the set S or Z, respectively. It is enough to satisfy all clauses of the first kind. Clauses $(x_Z \le x_S \vee x_Z \le x_e)$ are satisfied since the greatest element of Z either belongs to S or is equal to e. Because $x_k \in Z$, the variable x_Z is set to be greater than or equal to x_1. Therefore the last clause is also satisfied. □

Consider the following forms of filters.

$$z_1 \approx \dots \approx z_k \prec y_1 \tag{3.1}$$

$$z_1 \prec y_1 \approx \dots \approx y_k \tag{3.2}$$

We say that a preorder $\preceq$ with domain $Dom(\preceq) = \{x_1, \dots, x_n\}$ is of the form, let's say, (3.1) if $n = k + 1$ and $x_{i_1} \approx \dots \approx x_{i_{n-1}} \prec x_{i_n}$ for some permutation of $\{x_1, \dots, x_n\}$.

The next theorem shows us the difference between positive temporal languages with non-unary and only unary polymorphisms. This difference is expressed using their filters.

Theorem 48. *Let Γ be a positive temporal language. Consider the following conditions.*

1. *All filters minimal with respect to $\ll$ in Γ are of the form (3.1).*
2. *All filters minimal with respect to $\ll$ in Γ are of the form (3.2).*

If neither of these conditions holds, then sPol(Γ) contains only essentially unary polymorphisms.

For the moment we leave this theorem without proof, which is presented in the next section. Here, we show that the expressive power of a positive temporal pattern satisfying item 1 of Theorem 48 is not higher than the one of $(x_1 \le x_2 \vee x_1 \le x_3)$. The similar statement concerning languages satisfying item 2 of Theorem 48 and $(x_2 \le x_1 \vee x_3 \le x_1)$ is also true.

First we give some preliminary results.

Lemma 49. *Let $R(x_1, x_2)$ be a positive temporal relation with a filter $x_2 \prec x_1$. Moreover, assume that $\mathcal{F}(R)$ does not contain $x_1 \prec x_2$. Then $R(x_1, x_2)$ is equal to $(x_1 \le x_2)$.*

Proof. There are only three possible filters for a relation with two variables. By assumption, $x_1 \prec x_2$ is not a filter. By Corollary 40, a preorder $x_1 \approx x_2$ is not a filter. Therefore $x_2 \prec x_1$ is the only filter and hence R is equal to $(x_1 \le x_2)$. □

Lemma 50. *Let $R(x_1, x_2, x_3)$ be a positive temporal relation with a filter $x_1 \approx x_2 \prec x_3$. Moreover assume that $x_3 \prec x_1 \approx x_2$ does not belong to $\mathcal{F}(R)$. Then $(v_1 \le v_2 \vee v_1 \le v_3) \in [R]$.*

Proof. (Part One) First, we show that $(v_1 \leq v_2) \in [R]$. If either $x_3 \prec x_1$ or $x_3 \prec x_2$ is a filter of R, then, as we claim, either $R_1(x_1, x_3) := (\exists x_2\ R(x_1, x_2, x_3))$ is equivalent to $(x_1 \leq x_3)$ or $(\exists x_1\ R(x_1, x_2, x_3))$ is equivalent to $(x_2 \leq x_3)$. Consider the first case. By Lemma 35, the preorder $x_3 \prec x_1$ is a filter of R_1. To use Lemma 49, we need to show that $\mathcal{F}(R_1)$ does not contain $x_1 \prec x_3$. Assume the contrary. Then, by Lemma 35, the preorder $x_1 \prec x_3$ belongs also to $\mathcal{F}(R)$. But then $x_1 \approx x_2 \prec x_3$ cannot be a filter of R. The case where $x_3 \prec x_2$ is a filter is analogous.

If neither $x_3 \prec x_1$ nor $x_3 \prec x_2$ is a filter of R, then, as we show, the relation $R_2(x_2, x_3) := (R(x_1, x_2, x_3) \wedge x_1 = x_2)$ is equal to $(x_3 \leq x_2)$. Since $(x_1 \approx x_2 \prec x_3) \in \mathcal{F}(R)$, by lemmas 33 and 43, the preorder $x_2 \prec x_3$ is a filter of R_2. To use Lemma 49, we need to show that $x_3 \prec x_2$ is not a filter of R_1. Assume the contrary. Then, by Lemma 32, the preorder $x_3 \prec x_2 \approx x_1$ is a bound of R. By the assumptions of the lemma, it is not a filter of R. Moreover, neither $x_3 \prec x_1$ nor $x_3 \prec x_2$ belongs to $\mathcal{F}(R)$. The only remaining candidate for a filter more general than a bound $x_3 \prec x_1 \approx x_2$ is a preorder $x_1 \approx x_2$ but such a possibility is excluded by Corollary 40.

(Part Two) We now consider the relation $R_3(x_1, x_2, x_3) := (R(x_1, x_2, x_3) \wedge x_1 \leq x_3 \wedge x_2 \leq x_3)$. Note that by (Part One) we have that $R_3 \in [R]$. We describe the form of all tuples in R_3. Because of the second and the third conjunct, it is enough to focus on tuples of R satisfying $x_1 \leq x_3$ and $x_2 \leq x_3$. Since $x_1 \approx x_2 \prec x_3$ is a minimal bound of R, it contains a tuple q such that $q(x_1) < q(x_3)$, but it cannot be: $q(x_2) = q(x_1) < q(x_3)$. Moreover, Lemma 39 excludes the following possibilities: $q(x_1) < q(x_2) < q(x_3)$ and $q(x_2) < q(x_1) < q(x_3)$. Therefore, q must be of the form $q(x_1) < q(x_3) \leq q(x_2)$. Because every tuple in R_3 satisfies $x_2 \leq x_3$, the only possibility is: $q(x_1) < q(x_3) = q(x_2)$. Similarly, we can show that concerning tuples that satisfy $x_2 < x_3$, only those that satisfy $x_2 < x_3 = x_1$ belong to R_3.

(Part Three) Since R_3 is closed under all weakly increasing functions, R_3 is equal to $(x_1 \leq x_3 = x_2 \vee x_2 \leq x_3 = x_1)$. Finally, note that $\exists z\ R_3(v_2, v_3, z) \wedge v_1 \leq z$ is equivalent to $\exists z\ (v_2 \leq z = v_3 \vee v_3 \leq z = v_2) \wedge v_1 \leq z$ and hence to $(v_1 \leq v_2 \vee v_1 \leq v_3)$. ☐

Lemma 51. *Let Γ be a positive temporal language. If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form (3.1) and at least one of them has a domain of size greater than or equal to 3, then $(x_1 \leq x_2 \vee x_1 \leq x_3)$ has a qc-definition over Γ.*

Proof. (Part One) Let a preorder $\preceq_L$ of the form $x_1 \approx_L \ldots \approx_L x_{n-1} \preceq_L x_n$ where $n \geq 3$ be a filter of Γ. Let $R(x_1, \ldots, x_n, w_1, \ldots, w_k)$ be a relation from Γ such that $\preceq_L \in \mathcal{F}(R)$. First, define $R_1(x_1, \ldots, x_n)$ by $\exists w_1 \ldots \exists w_k\ R(x_1, \ldots, x_n, w_1, \ldots, w_k)$. By Lemma 35, the relation R_1 inherits the filter $\preceq_L$. Further, let $R_2(x_{n-2}, x_{n-1}, x_n)$ be equivalent to $\exists x_1 \ldots \exists x_{n-3}\ R_1(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n-3} x_i = x_{i+1}$. By Lemma 33, the set $\mathcal{F}(R_2)$ contains either a filter $x_{n-2} \approx x_{n-1} \prec x_n$ or a filter $x_{n-1} \prec x_n$. In the latter case, by lemmas 32 and 35, R contains a bound $\preceq_R$ of the form $x_{n-1} \prec_R x_n \approx_R x_1 \approx_R \ldots \approx_R x_{n-2}$. So, the relation R must have some filter more general than $\preceq_R$. Of course, it cannot be a filter of the form (3.2) for $k > 1$. It cannot be also an empty filter, or a filter of the form $y_1 \approx \ldots \approx y_k$. The remaining possibility is that it is a filter of the form $x_{n-1} \prec x_i$ where $i \in \{1, \ldots, n-2, n\}$. We now exclude this possibility. Since $\preceq_L \in \mathcal{F}(R)$, the preorder $x_{n-1} \preceq x_n$ cannot be a filter of R. Assume now on the contrary that $x_{n-1} \prec x_i$ for some $1 \leq i \leq n-2$ is a filter of R. Because $\preceq_L$ is a filter of R, there exists a tuple $t = \langle q(x_1), \ldots, q(x_n) \rangle \in R$ compatible with $x_1 \approx \ldots \approx x_{i-1} \approx x_{i+1} \approx \ldots \approx x_{n-1} \prec x_n$. Moreover, by Lemma 39, $q(x_i)$ cannot be strictly less than $q(x_n)$. Therefore t is compatible with $x_{n-1} \prec x_i$ and hence this preorder is not a filter of R. Concluding, the preorder $x_{n-2} \approx x_{n-1} \prec x_n$ is a filter of R_2.

In the following ((Part Two)–(Part Four)) we prove that $x_n \prec x_{n-1} \approx x_{n-2}$ is not a filter of R_2.

(Part Two) We first show that if $x_n \prec x_{n-1}$ is a bound of R, then $x_n \prec x_{n-1}$ is a bound of R_2. Assume the contrary. Then there exists a tuple $\langle q(x_{n-2}), q(x_{n-1}), q(x_n)\rangle$ in R_2 compatible with $x_n \prec x_{n-1}$. By the definition of R_2, the relation R contains a tuple $\vec{t} = \langle q_1(x_1), \ldots, q_1(x_n), q_1(w_1), \ldots, q_1(w_k)\rangle$ such that $q_1(x_i) = q_1(x_{i+1})$ for $i = 1, \ldots, n-3$ and $q_1(x_i) = q(x_i)$ for $i = n-2, n-1, n$. Note that $\vec{t}$ is compatible with $x_n \prec x_{n-1}$. Therefore $x_n \prec x_{n-1}$ is not a bound of R and hence we have a contradiction.

(Part Three) Now we claim that if $x_n \prec x_i$ for some $1 \leq i \leq n-2$ is a bound of R, then $x_n \prec x_{n-2}$ is a bound of R_2. Assume the contrary. Then there exists a tuple $\langle q(x_{n-2}), q(x_{n-1}), q(x_n)\rangle \in R_2$ compatible with $x_n \prec x_{n-2}$. By the definition of R_2, the relation R contains a tuple $\vec{t} = \langle q_1(x_1), \ldots, q_1(x_n), q_1(w_1), \ldots, q_1(w_k)\rangle$ such that $q_1(x_i) = q_1(x_{i+1})$ for $i = 1, \ldots, n-3$ and $q_1(x_i) = q(x_i)$ for $i = n-2, n-1, n$. Note that $\vec{t}$ is compatible with $x_n \prec x_i$ for all $1 \leq i \leq n-2$. Therefore $x_n \prec x_i$ cannot be a bound of R and hence we have a contradiction.

(Part Four) Now, we turn to prove that $x_n \prec x_{n-2} \approx x_{n-1}$ is not a filter of R_2. Assume the contrary. Then by lemmas 32 and 35 a preorder $\preceq_{R1}$ of the form $x_n \prec_{R1} x_1 \approx_{R1} x_2 \approx_{R1} \ldots \approx_{R1} x_{n-1}$ is a bound of R. Therefore, there must be some filter of R more general than $\preceq_{R1}$. Again, it cannot be a filter of the form (3.2) where $k > 1$, neither a filter of the form $y_1 \approx \ldots \approx y_l$ nor an empty filter. Now, we handle the situation where $x_n \prec x_i$ for some $1 \leq i \leq n-1$ is a filter of R. If $i = n-1$, then, by (Part Three), the preorder $x_n \prec x_{n-1}$ is a bound of R_2. Therefore $x_n \prec x_{n-2} \approx x_{n-1}$ is not a filter of R_2 and hence we have a contradiction. If $i < n-1$, then by (Part Four), the preorder $x_n \prec x_{n-2}$ is a bound of R_2, but it again contradicts the fact that $x_n \prec x_{n-2} \approx x_{n-1}$ is a filter of R_2.

(Part Five) We established so far that $x_{n-2} \approx x_{n-1} \prec x_n$ is a filter of R_2 but $x_n \prec x_{n-2} \approx x_{n-1}$ is not. Now, we can use Lemma 50 to express $(v_1 \leq v_2 \vee v_1 \leq v_3)$ over R_2 and hence over Γ. □

Lemma 52. *Let Γ be a positive temporal language.*

1. *If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form $z_1 \prec y_1$, then $[\Gamma] \subseteq [x_1 \leq x_2]$.*
2. *If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form (3.1) and at least one of them has a domain of size greater than or equal to 3, then $[\Gamma] = [\Gamma_{Left}]$.*
3. *If all minimal with respect to $\ll$ filters in $\mathcal{F}(\Gamma)$ are of the form (3.2) and at least one of them has a domain of size greater than or equal to 3, then $[\Gamma] = [\Gamma_{Right}]$.*
4. *If $[\Gamma] \subseteq [x_1 \leq x_2]$, then either $[\Gamma]$ is equal to $[x_1 \leq x_2]$ or to $[x_1 = x_2]$.*

Proof. (Part One) First observe that if all filters of Γ are of the form $z_1 \prec y_1$, then $[\Gamma] \subseteq [x_1 \leq x_2]$. Simply, if the filters of a relation R are $z_i \prec y_i$ for $i = 1, \ldots, k$, then $\bigwedge z_i \geq y_i$ is a qc-definition of R. The definition is correct since $z_i \geq y_i$ and $z_i \prec y_i$ forbid in fact the same configuration of variables z_i, y_i.

(Part Two) Because of lemmas 51 and 47, it is enough to prove that $[\Gamma] \subseteq [\Gamma_{Left}]$. Take any atom $R(x_1, \ldots, x_n)$ from Γ. A definition of R in Γ_{Left} contains a conjunct $R^{k+1}_{Left}(x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}})$ for each filter $x_{i_2} \approx \ldots \approx x_{ik+1} \prec x_{i_1}$ of the form (3.1).

We now show that the definition is indeed correct. Let $R \in \Gamma$ and ϕ_R be our definition of R in Γ_{Left}. We first prove that if a tuple $\vec{t}$ does not satisfy ϕ_R, then it is compatible with some bound of R. Indeed, if $\vec{t}$ does not satisfy some conjunct R^{k+1}_{Left} of ϕ_R, then it is compatible with some filter less flat than $x_{i_2} \approx \ldots \approx x_{i_{k+1}} \prec x_{i_1}$. Conversely, if $\vec{t}$ is not in R, then t is compatible with some bound that is less flat than some filter $x_{i_2} \approx \ldots \approx x_{i_{k+1}} \prec x_{i_1}$. Then $q(x_{i_1})$ is greater than all of $q(x_{i_2}), \ldots, q(x_{i_{k+1}})$ and a projection of $\vec{t}$ to $\{x_{i_1}, \ldots, x_{i_{k+1}}\}$ does not belong to R^{k+1}_{Left}.

(Part Three) The proof of item 3 is dual to the proof of item 2.

(Part Four) Now we prove item 4 of the lemma, which concerns relations qc-definable over $x_1 \leq x_2$. All filters of such relations are of the form $y \prec z$. In fact we show the following. If there exists at least two variables $x_i, x_j \in Dom(\prec)$ for some $\prec \in \mathcal{F}(\Gamma)$ such that $x_i \prec x_j$ and there is no bound $\preceq_1$ such that $x_j \prec_1 x_i$, then $[\Gamma] = [x_1 \leq x_2]$. Otherwise $[\Gamma] = [x_1 = x_2]$. Let $R(x_1, x_2, \ldots, x_n)$ be in Γ and $x_i \prec x_j$ for $i, j \in \{1, \ldots, n\}$ be a filter of R. Let $W = \{x_1, \ldots, x_n\} \setminus \{x_i, x_j\}$. If there is no $x_j \prec x_i$ in $\mathcal{F}(R)$, then, by Lemma 49, the relation $\exists W\ R(x_1, \ldots, x_n)$ defines $x_j \leq x_i$.

If for every filter $x_i \prec x_j$ there is a filter $x_j \prec x_i$, then we can define R in $[x_1 = x_2]$ by replacing each such pair of filters by a conjunct $x_i = x_j$. In turn, the relation $(x_1 = x_2)$ may be defined by a formula $\exists x_3 \ldots \exists x_n\ R(x_1, \ldots, x_n)$ where x_1, x_2 are two variables linked by filters $x_1 \prec x_2$ and $x_2 \prec x_1$. □

Proof of Theorem 44. (Part One) Consider the following sets of relations:

1. $[\Gamma_{Left}]$
2. $[\Gamma_{Right}]$
3. $[x_1 \leq x_2]$
4. $[x_1 = x_2]$

By Theorem 48, it is enough to prove that if all filters minimal with respect to $\ll$ of Γ are either of the form (3.1) or of the form (3.2), then $[\Gamma]$ is equal to exactly one of the above classes. By Lemma 52, we obtain that $[\Gamma]$ is equal to at least one of them. To show that these classes are pairwise disjoint we use Theorem 20 and appropriate polymorphisms from the preceding lemmas. By Lemma 45, we have that the first of the above families is closed under *dual-spp* operation but is not preserved by *spp*. By Lemma 46, it is not closed under any surjective oligopotent QNU polymorphism. Using the same lemmas we obtain a similar statement about the second family. Further, from [7] we know that the third family is closed under a surjective oligopotent QNU polymorphism. To distinguish the third and the fourth of the above sets using a function (in fact a permutation of rational numbers) note that the former is not closed under any strictly decreasing unary function.

(Part Two) Since all the classes are pairwise disjoint, we can infer from Lemma 52 that $[\Gamma] = [\Gamma_{Left}]$ if and only if all filters from Γ are of the form (3.1) and at least one of them has a domain greater than or equal to 3. Using similar reasoning as in (PartTwo) of the proof of Lemma 52, we can infer that each relation from Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$.

Similarly we can prove that if $[\Gamma] = [\Gamma_{Right}]$, then each relation from Γ may be defined by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$.

Further, if $[\Gamma]$ is equal to $[x_1 \leq x_2]$, then all filters are of the form $z \prec y$. As in (Part One) of the proof of Lemma 52, we can show that each relation in Γ is definable by a conjunction of inequalities.

Finally, using (Part Four) of the proof of Lemma 52 we can show that if $[\Gamma]$ is equal to $[x_1 = x_2]$, then each relation in Γ may be defined as a conjunction of equalities. In this case Γ is closed under all surjections. □

3.5 Filters of Hard Relations

Although the previous section contains the proof of Theorem 44, Theorem 48 was left without an explanation. This section is devoted to fill this hole.

The idea behind the proof is to show that if a positive temporal template Γ contains a filter that is not of the form (3.1) and a filter that is not of the form (3.2), then $[\Gamma]$ contains some positive non-negative equality relation R. By Corollary 25, the relation R and hence, by Theorem 20, the language Γ, is closed under essentially unary polymorphisms only. To prove this we consider a few cases. In most of them, we use the following lemma.

If an n-ary (at least ternary) positive temporal relation $R(x_1, \ldots, x_n)$ is different from $\mathbb{Q}^n$ and contains the relation $\bigvee_{i \neq j} x_i = x_j$ where $1 \leq i, j \leq n$ as a subrelation then we call it *potentially positive non-negative*.

Lemma 53. *Let R be a potentially positive non-negative relation. Then it is closed under essentially unary polymorphisms only.*

Proof. Define R_u as $\bigwedge_{\Pi \in S(n)} R(x_{\Pi(1)}, \ldots, x_{\Pi(n)})$, where S_n is the set of all permutations on n elements. Since R is potentially positive non-negative, all tuples $\langle q(x_1), \ldots, q(x_n) \rangle$ from $\mathbb{Q}^n \setminus R$ satisfy $q(x_{i_1}) < \ldots < q(x_{i_n})$ for some permutation $i_1, \ldots, i_n$. Because in $\mathbb{Q}^n \setminus R$ there is at least one such tuple, the relation R_u contains no such tuple. Therefore $R_u(x_1, \ldots, x_n)$ is equal to $\bigvee_{i \neq j} x_i = x_j$. Since this definition of R_u is in a reduced form and contains at least three variables, by Lemma 24, the relation R_u is positive and non-negative. Hence by, Corollary 25, it is closed only under essentially unary polymorphisms. Since $R_u \in [R]$, by Theorem 20, we have that R is closed under essentially unary polymorphisms only. □

Consider the followings forms of filters.

$$x_1 \approx \ldots \approx x_k \prec y_1 \approx \ldots \approx y_l \tag{3.3}$$

$$x_1^1 \approx \ldots \approx x_{l_1}^1 \prec \ldots \prec x_1^n \approx \ldots \approx x_{l_n}^n \tag{3.4}$$

If $\mathcal{F}(\Gamma)$ contains a filter than is not of the form (3.1) and a filter that is not of the form (3.2), then one of the following cases holds.

1. The language Γ contains a filter of the form (3.3) where $k > 1$ and $l \geq 1$ as well as a filter of the same form where $l > 1$ and $k \geq 1$, or

2. there exists a filter of the form (3.4) for $n \geq 3$.

The next two sections handles these cases. The section 3.5.1 covers the first situation. The second case is taken care of by the section 3.5.2.

3.5.1 Filters Of Range 2

Here we prove the following.

Proposition 54. *Let Γ be a positive temporal template with filters $\preceq_L$ and $\preceq_R$ defined as follows. The preorder $\preceq_L$ is of the form (3.3) with $k > 1$ and $l \geq 1$. The preorder $\preceq_R$ is also of the same form (3.3), but with $l > 1$ and $k \geq 1$. Then sPol(Γ) contains only essentially unary polymorphisms.*

Our strategy here is to show first that 'short' relations with 'short' filters of the form (3.3) with $k > 1$ and $l \geq 1$ are closed only under surjective unary polymorphisms or they express $(x_1 \leq x_2 \vee x_1 \leq x_3)$. Afterward, we consider arbitrary relations with arbitrary long filters. For a long relation R_L, we show that it can express a simple relation R_S. Therefore R_L can express everything expressible by R_S; equivalently, R_L cannot have more surjective polymorphisms than R_S. Lemmas 35, 33, and 43 ensure that R_S obtained from R_L has an appropriate filter. Short relations have either arity 3 or 4. The first case is handled by Corollary 56, the second case by Lemma 57. See the example at the end of this subsection.

Lemma 55. *Let $R(x_1, x_2, x_3)$ be a positive temporal relation with filters $x_1 \approx x_2 \prec x_3$ and $x_3 \prec x_1 \approx x_2$. Then sPol$(\Gamma)$ contains only essentially unary polymorphisms.*

Proof. (Part One) We first establish possible tuples of R. Note that there is no bound of R more general than $x_1 \approx x_2 \prec x_3$. Therefore there is a tuple $\langle q(x_1), q(x_2), q(x_3)\rangle \in R$ such that $q(x_1) < q(x_3)$. Similarly as in the proof of Lemma 50, we infer that all tuples in R satisfying that condition are of the following form: $q(x_1) < q(x_3) \leq q(x_2)$. In the same way, we show that there exists a tuple such that $q(x_2) < q(x_3)$ and it must be of the form: $q(x_2) < q(x_3) \leq q(x_1)$. Since R is closed under all weakly increasing functions, it contains the relation $(x_1 < x_3 = x_2 \vee x_2 < x_3 = x_1)$.

Using $x_3 \prec x_1 \approx x_2$, we can show that there are tuples satisfying $x_3 < x_1$ and they are of the form: $q(x_2) \leq q(x_3) < q(x_1)$. Finally, there are tuples satisfying $x_3 < x_2$ and their form is as follows: $q(x_1) \leq q(x_3) < q(x_2)$. Therefore R contains $x_2 = x_3 < x_1 \vee x_1 = x_3 < x_2$.

Note that R is not uniquely determined. Intuitively, a relation R, independently on the exact form, says that x_3 must be somewhere between x_1 and x_2, that is, either $q(x_1) \leq q(x_3) \leq q(x_2)$ or $q(x_2) \leq q(x_3) \leq q(x_1)$ holds. Moreover both these situations are always contained in R. All possibilities where x_3 is not between x_1 and x_2 are excluded by Lemma 39.

(Part Two) Here, we show that $R_1(y_1, y_2, y_3) := \exists z\ R(y_1, z, y_2) \wedge R(y_1, z, y_3)$ is a potentially positive non-negative relation. Then we use Lemma 53. To see that $R_1 \neq \mathbb{Q}^3$, observe that R_1 does not contain any tuple satisfying $q(y_2) < q(y_1) < q(y_3)$. What remains to be shown is that $R_p(y_1, y_2, y_3) := (\bigvee_{i \neq j} y_i = y_j)$, where $1 \leq i, j \leq 3$, is a subrelation of R_1.

We claim that for each tuple of R_p there is a place for z in R_1. If such a tuple satisfies $q(y_1) = q(y_2)$, then we set z to $q(y_3)$. In this case $q(y_2)$ as well as $q(y_3)$ is between $q(y_1)$ and z. In all other cases we set z to $q(y_2)$. □

As an immediate consequence of lemmas 55 and 50 we get the following.

Corollary 56. *Let $R(x_1, x_2, x_3)$ be a positive temporal relation with a filter $x_1 \approx x_2 \prec x_3$. Then either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [\Gamma]$ or sPol(Γ) contains only essentially unary polymorphisms.*

Lemma 57. *Let $R(x_1, x_2, x_3, x_4)$ be a positive temporal relation with a filter $x_1 \approx x_2 \approx x_3 \prec x_4$. Then either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$ or sPol(R) contains only essentially unary polymorphisms.*

Proof. In the following, whenever we write w_i, for some i, it is one of $\{x_1, x_2, x_3, x_4\}$.

(Part One) Consider the situation where $\mathcal{F}(R)$ contains a filter of the form $w_1 \approx w_2 \prec w_3$. By Lemma 35, a relation $\exists w_4 R(x_1, x_2, x_3, x_4)$ inherits this filter. Then we are done by Corollary 56. In turn, if $\mathcal{F}(R)$ contains a filter $w_1 \approx w_2 \prec w_3 \approx w_4$, then, by lemmas 33 and 43, the relation $R(x_1, x_2, x_3, x_4) \wedge w_3 = w_4$ contains a filter $w_1 \approx w_2 \prec w_3$. Now, it is enough to use Corollary 56 for it.

(Part Two) Define $R_1(x_1, x_3, x_4)$ as $R(x_1, x_2, x_3, x_4) \wedge x_2 = x_3$. Because $(x_1 \approx x_2 \approx x_3 \prec x_4) \in F(R)$, by Lemma 33, we have that either filter of the form $x_1 \approx x_3 \prec x_4$ or $x_1 \prec x_4$ belongs to $\mathcal{F}(R_1)$. If the former holds, then we are done by Corollary 56. Otherwise, by Lemma 32, the relation R has a bound $\preceq_1$ of the form $x_1 \prec_1 x_4 \approx_1 x_2 \approx_1 x_3$. A similar reasoning can be used with respect to relations $R_2(x_2, x_3, x_4)$ defined by $R(x_1, x_2, x_3, x_4) \wedge x_1 = x_3$ and $R_3(x_2, x_3, x_4)$ equivalent to $R(x_1, x_2, x_3, x_4) \wedge x_1 = x_2$. Concluding this part, we are either done or we can assume that $\mathcal{B}(R)$ contains $\preceq_1$ as well as $\preceq_2$ of the form $x_2 \prec_2 x_4 \approx_2 x_1 \approx_2 x_3$ and $\preceq_3$ given by $x_3 \prec_3 x_4 \approx_3 x_1 \approx_3 x_2$.

(Part Three) Here, we claim that if R contains any filter of the form $w_1 \prec w_2$, then $(v_1 \leq v_2) \in [R]$. In the case of $w_1 = x_4$ and $w_2 = x_1$, we consider a relation $R_1(x_1, x_4) = \exists x_2 \exists x_3\, R(x_1, x_2, x_3, x_4)$. By Lemma 35, the set $\mathcal{F}(R_1)$ contains $x_4 \prec x_1$. To use Lemma 49, it is enough to prove that $\mathcal{F}(R_1)$, or in fact, by Lemma 35, $\mathcal{F}(R)$ does not contain $x_1 \prec x_4$. It holds since $x_1 \approx x_2 \approx x_3 \prec x_4$ is a filter of R. We reason in a similar way if either $w_2 = x_2$ or $w_2 = x_3$. The last case is for $w_1 = x_i$ and $w_2 = x_j$ where $i, j = 1, 2, 3$. As we show, the existence of such a filter contradicts earlier assumptions. Focus on $x_1 \prec x_2$. Because $x_1 \approx x_2 \approx x_3 \prec x_4$ is a filter of R, there exists a tuple $q : \{x_1, x_2, x_3, x_4\} \to \mathbb{Q}$ satisfying $q(x_1) = q(x_3) < q(x_4)$. By Lemma 39, we infer that $q(x_2)$ must be greater than or equal to $q(x_4)$. Since q is compatible with $x_1 \prec x_2$, this preorder cannot be a filter of R.

(Part Four) This paragraph is devoted to the case where $\mathcal{F}(R)$ contains a filter of the form $w_1 \prec w_2 \approx w_3$. In this case, as we show, the relation $(v_1 \leq v_2)$ belongs to $[R]$, or we are done by Corollary 56. Focus on an arbitrary $w_1 \prec w_2 \approx w_3$. Consider $R_1(w_1, w_3)$ equivalent to $\exists w_4 \exists w_2\, R(x_1, x_2, x_3, x_4) \wedge w_2 = w_3$. By lemmas 35, 33, and 43, this relation has a filter $w_1 \prec w_3$. As we show, the set $\mathcal{F}(R_1)$ does not contain $w_3 \prec w_1$ and then we use Lemma 49. Assume on the contrary that $w_3 \prec w_1$ is a filter of R_1. Then by lemmas 35 and 32, the preorder $w_2 \approx w_3 \prec w_1$ is a bound of R. But if $\mathcal{B}(R)$ contains some $\preceq$, then there must be some filter in $\mathcal{F}(R)$ more general than $\preceq$. There are three possible forms of preorders more general than $w_2 \approx w_3 \prec w_1$. Two of them are covered by (Part One) and (PartThree). The third one is excluded by Corollary 40.

(Part Five) In the following we consider the situation where $(w_1 \leq w_2) \in [R]$. From (Part Two) we know that $\mathcal{B}(R)$ contains: $\preceq_i$ for $i = 1, 2, 3$. Consider a relation $R_1(x_1, x_2, x_3, x_4) = R(x_1, x_2, x_3, x_4) \wedge x_1 \leq x_4 \wedge x_2 \leq x_4 \wedge x_3 \leq x_4$. Now, we focus on its tuples. It is clear that each tuple of R_1 satisfies $q(x_i) \leq q(x_4)$ where $i = 1, 2, 3$. Since $\preceq_1 \in \mathcal{B}(R)$, from Lemma 39 we infer that R does not contain a tuple where x_1 has the least value among all the variables. Similarly, because of $\preceq_2$ and $\preceq_3$, there are no tuples such that either x_2 or x_3 has the least value. By $x_1 \approx x_2 \approx x_3 \prec x_4$, the variable x_4 cannot be strictly greater than all remaining variables. Therefore R_1 is contained in $(x_2 = x_3 \leq x_4 = x_1 \vee x_1 = x_3 \leq x_2 = x_4 \vee x_1 = x_2 \leq$

$x_3 = x_4$). But $x_1 \approx x_2 \approx x_3 \prec x_4$ is a filter of R and hence R_1 contains a tuple $q(x_2) = q(x_3) < q(x_4) = q(x_1)$ as well as $q(x_1) = q(x_3) < q(x_4) = q(x_2)$ and $q(x_1) = q(x_2) < q(x_4) = q(x_3)$. Hence R_1 contains $(x_2 = x_3 \leq x_4 = x_1 \vee x_1 = x_3 \leq x_2 = x_4 \vee x_1 = x_2 \leq x_3 = x_4)$.

Now, observe that a relation $R_2(x_2, x_3, x_4) := \exists x_1\ R_1(x_1, x_2, x_3, x_4) \wedge x_1 \leq x_2 \wedge x_1 \leq x_3$ is equivalent to $(x_4 \leq x_2 \vee x_4 \leq x_3) \wedge x_2 \leq x_4 \wedge x_3 \leq x_4$. Finally, the relation $(v_1 \leq v_2 \vee v_1 \leq v_3)$ may be defined by $\exists z\ R_2(v_2, v_3, z) \wedge v_1 \leq z$.

(Part Six) What remains to be considered is the situation where $(v_1 \leq v_2) \notin [R]$. In this case, by (Part One), we can assume that R has neither $w_1 \approx w_2 \prec w_3$ nor $w_1 \approx w_2 \prec w_3 \approx w_4$ as a filter. From (Part Three) there are no filters of the form $w_1 \prec w_2$; and from (Part Four) no filters of the form $w_1 \prec w_2 \approx w_3$. In that case bounds: $\preceq_1, \preceq_2$, and $\preceq_3$, introduced in (Part Three), are indeed filters. Moreover, all filters are in one of the two following forms. They either look like $w_1 \prec w_2 \approx w_3 \approx w_4$ or like $w_1 \approx w_2 \approx w_3 \prec w_4$. Note that $x_1 \approx x_2 \approx x_3 \prec x_4$ is of the second of allowed forms. Finally, consider a relation $R_p(x_1, x_2, x_3, x_4)$ defined as $\bigwedge_{\Pi \in S_4} R(x_{\Pi(1)}, x_{\Pi(2)}, x_{\Pi(3)}, x_{\Pi(4)})$ where S_4 is a set of all permutations of $\{1, 2, 3, 4\}$. Observe that R_p is equivalent to $(x_1 = x_2 \vee x_1 = x_3 \vee x_1 = x_4) \wedge (x_2 = x_1 \vee x_2 = x_3 \vee x_2 = x_4) \wedge (x_3 = x_1 \vee x_3 = x_2 \vee x_3 = x_4) \wedge (x_4 = x_1 \vee x_4 = x_2 \vee x_4 = x_3)$. This relation is of course positive. We will now show that it is not negative. By Lemma 24, it is enough to show that its definition is in a reduced form. Assume on the contrary that we can remove some disjunct from one of clauses and obtain another definition of R_p. Because the above definition of R_p is quite symmetric, we can assume without loss of generality that $x_1 = x_2$ was removed from the first clause. Let ϕ be a new formula without the removed conjunct. We will show that ϕ does not define R_p. Now, a valuation $q : \{x_1, x_2, x_3, x_4\} \to \mathbb{Q}$ satisfies ϕ if $q(x_1) = q(x_3)$ or $q(x_1) = q(x_4)$. Therefore ϕ in contrast to the original formula is not satisfied by $q_1(x_1) = q_1(x_2) < q_1(x_3) = q_1(x_4)$. Thus ϕ does not define R_p. Finally by Corollary 25, the relation R_p is preserved by essentially unary polymorphisms only. □

Lemma 58. *Let $R(x_1, \ldots, x_n)$ be a positive temporal relation. If R has any filter of the form (3.3) where $k > 1$ and $l \geq 1$, then either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$ or sPol(Γ) contains only essentially unary polymorphisms.*

Proof. (Part One) Let $\preceq_g$ of the form $y_1 \approx_g \ldots \approx_g y_k \preceq_g z_1 \approx_g \ldots \approx_g z_l$ be a filter of R where $k > 1$, $l \geq 1$, and all y_i, z_j are in $\{x_1, \ldots, x_n\}$. Let $W = \{x_1, \ldots, x_n\} \setminus \{y_1, \ldots, y_k, z_1, \ldots, z_l\}$. Define $R_1(y_1, \ldots, y_k, z_1, \ldots, z_l)$ as $\exists W\ R(x_1, \ldots, x_n)$. By Lemma 35, the relation R_1 inherits the filter $\preceq_g$. Further consider $R_2(y_1, \ldots, y_k, z_l)$ equivalent to $R_1(y_1, \ldots, y_k, z_1, \ldots, z_l) \wedge \bigwedge_{i=2}^{l-1} z_i = z_{i+1}$. By Lemma 33, we have that either $y_1 \approx_1 \ldots \approx_1 y_k$ or $y_1 \approx_2 \ldots \approx_2 y_k \preceq z_l$ is a filter of R_2. Note that Corollary 40 excludes the first possibility.

(Part Two) If $k > 3$, then we define $\exists y_1 \ldots \exists y_{k-3}\ R_3(y_{k-2}, y_{k-1}, y_k, z_l) = R_2(y_1, \ldots, y_k, z_l) \wedge \bigwedge_{i=1}^{k-3} y_i = y_{i+1}$. By Lemma 33, the set $\mathcal{F}(R_3)$ contains either $y_{k-2} \approx y_{k-1} \approx y_k \prec z_l$ or $y_{k-1} \approx y_k \prec z_l$. If the first case holds, then we use here Lemma 57, otherwise, we use Corollary 56 for a relation $\exists y_{k-2}\ R_3(y_{k-2}, y_{k-1}, y_k, z_l)$. In both cases we are done.

If $k = 2$ or $k = 3$, then we use a similar reasoning directly to R_2. □

Likewise, we can show that if a positive temporal relation has any filter of the form 3.3 where $k \geq 1$ and $l > 1$ then either it is closed only under essentially unary surjective polymorphisms or it can express $(x_2 \leq x_1 \vee x_3 \leq x_1)$. The following statement in fact finishes the proof of Proposition 54.

Lemma 59. *If both* $(v_1 \leq v_2 \vee v_1 \leq v_3)$ *and* $(v_2 \leq v_1 \vee v_3 \leq v_1)$ *belong to* $[\Gamma]$*, then* Γ *is closed only under essentially unary polymorphisms.*

Proof. Note that $(x_1 \leq x_2 \vee x_1 \leq x_3) \wedge (x_2 \leq x_1 \vee x_3 \leq x_1)$ contains both: a filter $x_1 \prec x_2 \approx x_3$ and $x_2 \approx x_3 \prec x_1$. Therefore, the result follows by Lemma 55. □

Proof of Proposition 54. Let R be a relation of Γ with the filter $\preceq_L$. Then, by Lemma 58, either $(v_1 \leq v_2 \vee v_1 \leq v_3) \in [R]$ or *sPol*(R) contains only essentially unary polymorphisms. Similarly, for some R_1 that has $\preceq_R$ as a filter we can show that either all its surjective polymorphisms are essentially unary or $(v_2 \leq v_1 \vee v_3 \leq v_1)$ belongs to $[R_1]$. To complete the proof we use Lemma 59. □

We finish this subsection with an example that illustrates Proposition 54.

Example 60. *Consider a positive relation* R_L *given by* $(x_1 \leq x_2 \vee x_1 \leq x_3) \wedge (x_5 \leq x_4 \vee x_6 \leq x_4) \wedge \phi(y_1, \ldots, y_m)$ *where* $\{x_1, \ldots, x_6\} \cap \{y_1, \ldots, y_m\} = \emptyset$*. It is straightforward to show that* $x_2 \approx x_3 \prec x_1$ *as well as* $x_4 \prec x_5 \approx x_6$ *are minimal wrt* $\ll$ *filters of* R *– see Example 41. We now claim that sPol*(Γ) *contains essentially unary polymorphisms only.*

Now, define R_S *as* $\exists y_1 \ldots \exists y_m \exists x_4 \exists x_5 \exists x_6 R_L(x_1, \ldots, x_6, y_1, \ldots, y_m)$ *By Lemma 35, we have that* R_S *inherits the filter* $x_2 \approx x_3 \prec x_1$*. From Lemma 50 we infer that* $(v_1 \leq v_2 \vee v_1 \leq v_3)$ *belongs to* $[R_S]$*. Hence it belongs to* $[R_L]$*; note that* R_S *is qc-definable in* R_L *– recall the definition of* R_S *above. Similarly we can show that* $(v_2 \leq v_1 \vee v_3 \leq v_1) \in R_L$*. By Lemma 59, we have that sPol*(R_L) *contains essentially unary polymorphisms only.*

3.5.2 Filters Of Range Greater Than 2

Here we prove the following.

Proposition 61. *Let* Γ *be a positive temporal language. If there exists any minimal with respect to* $\ll$ *filter in* $\mathcal{F}(\Gamma)$ *whose range is strictly greater than 2, then sPol*(Γ) *contains only essentially unary polymorphisms.*

The strategy here is similar to one in the preceding section. Here, 'short' filters are of the form (3.4) where each $l_i = 1$ for all $1 \leq i \leq n$. This time we consider a relation R_S to be 'short' if it contains a 'short' filter $\preceq_S$ and $Dom(R_S) = Dom(\preceq_S)$. 'Long' relations are arbitrary relations having filters of the form (3.4).

Concerning 'short' relations, we consider two situations: either a short relation R_S contains a filter of the range 2 or not. The former case is covered by Lemma 64, the latter one by Lemma 62.

Lemma 62. *Let* $R(x_1, \ldots, x_n)$ *be a positive temporal relation with a filter* $\preceq$ *of the form* $x_1 \prec \ldots \prec x_n$ *where* $n \geq 3$*. Assume that* $\preceq$ *is minimal with respect to the size of its range filter of* R*. Then sPol*(R) *contains only essentially unary polymorphisms.*

Proof. We just show how to use Lemma 53 here. Since $\mathcal{F}(R)$ contains $\preceq$, the relation R cannot be equal to $\mathbb{Q}^n$. Because $\preceq$ is of minimal range among all filters, all members of $\mathcal{F}(R)$ are of the form $x_{i_1} \prec \ldots \prec x_{i_n}$ for some permutation $i_1, \ldots, i_n$. Therefore R contains $\bigvee_{i \neq j} x_i = x_j$ where $1 \leq i, j \leq n$ as a subrelation. □

Example 63. *A simple instance of a positive temporal relation, which has a minimal wrt $\ll$ filter of range 3 is a relation R defined by a formula $(x_1 \leq x_2 \vee x_2 \leq x_3)$. Observe first that R has a filter $x_3 \prec x_2 \prec x_1$ and contains the relation $\bigvee_{i \neq j} x_i = x_j$ where $1 \leq i, j \leq 3$. Further, consider the relation $\bigwedge_{\Pi \in S_3}(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)})$ where S_3 is the set of all permutations on the set $\{1, 2, 3\}$. Note that this relation belongs to $[R]$ and is equal to $\bigvee_{i \neq j} x_i = x_j$ where $1 \leq i, j \leq 3$, which by Lemma 24 is positive and non-negative. Therefore by Corollary 25 and Thereom 20, we have that all surjective polymorphisms of R are essentially unary.*

Lemma 64. *Let $R(x_1, \ldots, x_n)$ be a positive temporal relation with a filter $\preceq$ of the form $x_1 \prec \ldots \prec x_n$ where $n \geq 3$. Let $\preceq$ be minimal wrt $\ll$ and minimal with respect to the size of the range among all filters of range strictly greater than 2. Assume that all filters of range 2, if there are any, are either of the form (3.1) or (3.2). Then sPol(R) contains only essentially unary polymorphisms.*

Proof. (Part one) If there are no filters of range 2, then the lemma reduces to Lemma 62. Therefore, we assume that there are some filters of range 2. In the following we write w_i for some i to denote a member of a set $\{x_1, \ldots, x_n\}$.

(Part Two) First, we show how to express $(v_1 \leq v_2)$. Without loss of generality, we assume that all filters of range 2 are of the form (3.1). Let a preorder $\preceq_1$ of the form $w_1 \approx_1 \ldots \approx_1 w_{k-1} \prec_1 w_k$ be such a filter. Let V be equal to $\{x_1, \ldots, x_n\} \setminus \{w_1, \ldots, w_k\}$. Then, by Lemma 35, the relation $R_1(w_1, \ldots, w_k) := \exists V\, R(x_1, \ldots, x_n)$ inherits the filter $\preceq_1$. Now assume that R_1 contains a filter of the form $w_k \prec w_i$ where $1 \leq i \leq k-1$. Focus on $w_k \prec w_1$. Because $\preceq_1$ is a filter of R_1, the set $\mathcal{F}(R_1)$ does not contain $w_1 \prec w_k$. Therefore, by Lemma 35, the relation $R_2(w_1, w_k)$ defined by $\exists w_2 \ldots \exists w_{k-1}\, R_1(w_1, \ldots, w_k)$ has a filter $w_k \prec w_1$ but does not have $w_1 \prec w_k$ as a filter. By Lemma 49 the relation $R_2(v_1, v_2)$ is equivalent to $(v_1 \leq v_2)$.

Now assume that none of preorders of the form $w_k \prec w_i$, where $1 \leq i \leq k-1$, is a filter of R_1. Further, by lemmas 33 and 43, a relation $R_3(w_{k-1}, w_k) = R_1(w_1, \ldots, w_k) \wedge \bigwedge_{i=1}^{k-2} w_i = w_{i+1}$ contains a filter $w_{k-1} \prec w_k$. To use Lemma 49, it is enough to show that $\mathcal{F}(R_3)$ does not contain $w_k \prec w_{k-1}$. Assume the contrary. Then, by lemmas 32 and 35, R contains a bound $\preceq_2$ of the form $w_k \prec_2 w_1 \approx_2 \ldots \approx_2 w_{k-1}$. Further, there must be some filter in $\mathcal{F}(R)$ more general than $\preceq_2$. By Corollary 40, it cannot be a filter more general than $w_1 \approx \ldots \approx w_{k-1}$. Because none of preorders of the form $w_k \prec w_i$ belongs to $\mathcal{F}(R_1)$, the filter must be of the form $w_k \prec w_{i_1} \approx \ldots \approx w_{i_l}$ for $l \geq 2$. But it contradicts the assumption that restricts filters of range 2 to just one possible form.

(Part Three) Consider $R_S^n(x_1, \ldots, x_n)$ equivalent to $R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n-1} x_i \leq x_{i+1}$. Define recursively $R_S^{i-1}(x_1, \ldots, x_{i-1})$ by $\forall y \exists z\, R_S^i(x_1, \ldots, x_{i-1}, z) \wedge y \leq z$.

We now prove by induction that for each $1 < i \leq n$ the relation $R_S^i(x_1, \ldots, x_i)$ is equivalent to $\neg(x_1 < \ldots < x_i) \wedge x_1 \leq \ldots \leq x_i$. In the base case we consider R_S^n. It is clear that all its tuples $\langle q(x_1), \ldots, q(x_n)\rangle$ satisfy $x_1 \leq \ldots \leq x_n$ where $q : \{x_1, \ldots, x_n\} \to \mathbb{Q}$. Because $x_1 \prec \ldots \prec x_n$ is in $\mathcal{F}(R)$, there is no tuple $q : \{x_1, \ldots, x_n\} \to Q$ satisfying $x_1 < \ldots < x_n$. Hence R_S^n is a subrelation of $\neg(x_1 < \ldots < x_n) \wedge x_1 \leq \ldots \leq x_n$. We now show the converse. Assume on the contrary that one of tuples of this relation does not belong to R_S^n. Then there is some bound $\preceq_1$ of R_S^n flatter than $x_1 \prec \ldots \prec x_n$. If $\preceq_1$ is a filter, then it is flatter than the filter $x_1 \prec \ldots \prec x_n$, which was supposed to be minimal wrt $\ll$. Otherwise, there is some filter $\preceq_2$ more general than $\preceq_1$. By Lemma 39 all $\preceq$ such that $\preceq_2 \ll \preceq$ are bounds of R_S^n. Note that among them there must be some preorder more general than $x_1 \prec \ldots \prec x_n$ but it contradicts the assumption that $x_1 \prec \ldots \prec x_n$ is a filter.

Now, we perform the induction step. We assume that $R_S^k(x_1, \ldots, x_k)$ is equivalent to $\neg(x_1 < \ldots < x_k) \wedge x_1 \leq \ldots \leq x_k$ for $k = i$ and we claim that such an equivalence holds also for $k = i - 1$. To see this, distinguish two cases. The first one is if we set $x_1, \ldots, x_{i-1}$ such that $q(x_1) < \ldots < q(x_{i-1})$. In this case, if y has a value strictly greater than $q(x_{i-1})$, then, by induction hypothesis, there is no place for z to satisfy $R_S^i(x_1, \ldots, x_{i-1}, z)$. On the other hand, if at least one x_l where $1 \leq l \leq i-2$ is equal to x_{l+1} we can always, that is, independently on the value of y, find a place for z. It is enough to set z to be strictly greater than maximum of the value of y and $q(x_{i-1})$.

(Part Four) Finally define $R_f(x_1, x_2, x_3)$ as $\exists z_1 \exists z_2 \; R_S^3(z_1, x_2, z_2) \wedge z_1 \leq x_1 \leq z_2 \wedge z_1 \leq x_3 \leq z_2$. What remains to be shown is that R_f satisfies the conditions of Lemma 53. First, we show that R_f does not contain a tuple satisfying $x_1 < x_2 < x_3$ and hence it is not equal to $\mathbb{Q}^3$. Assume the contrary. By the definition of R_S^3, a value of x_2 must be equal either to z_1 or to z_2. If x_2 is equal to z_1 then z_2 must be greater than or equal to x_2. Since x_1 must be between z_1 and z_2, it cannot be strictly less than x_2. On the other hand, if x_2 is equal to z_2, then z_1 must be less than or equal to x_2 and hence x_3 cannot be strictly greater than x_2. Concluding, the relation R_f cannot contain a tuple of the form $q(x_1) < q(x_2) < q(x_3)$ for any $q : \{x_1, x_2, x_3\} \to \mathbb{Q}$. Now, we prove that $R_p^3 := (\bigvee_{i \neq j} x_i = x_j)$ for $1 \leq i, j \leq 3$ is a subrelation of R_f. We simply show for each tuple of R_p^3 where to put the values of variables z_1 and z_2. If a tuple of R_p^3 is of the form $(q(x_2) \leq q(x_1) \wedge q(x_2) \leq q(x_3))$, then we set z_1 to $q(x_2)$ and z_2 strictly greater than all $q(x_i)$ for $i = 1, 2, 3$. Otherwise, we set z_2 to $q(x_2)$ and z_1 strictly less than all of x_i. □

Proof of Proposition 61. Let $R(x_1, \ldots, x_n)$ be a relation from Γ. Let $w_1^1 \approx_1 \ldots \approx_1 w_{l_1}^1 \preceq_1 \ldots \preceq_1 w_1^k \approx_1 \ldots \approx_1 w_{l_k}^k$ where $k \geq 3$ be a filter of R. Let $W = \{x_1, \ldots, x_n\} \setminus \{w_1^1, \ldots, w_{l_1}^1, \ldots, w_1^k, \ldots, w_{l_k}^k\}$. First consider the following relation: $R_1(w_1^1, \ldots, w_{l_1}^1, \ldots, w_1^k, \ldots, w_{l_k}^k)$ defined by $\exists W \; R(x_1, \ldots, x_n)$. By Lemma 35, the relation R_1 inherits $\preceq_1$ from R. Now, we focus on $R_2(w_1^1, \ldots, w_1^k)$ that is equivalent to $R_1(w_1^1, \ldots, w_{l_1}^1, \ldots, w_1^k, \ldots, w_{l_k}^k) \wedge \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{l_i-1} w_j^i = w_{j+1}^i$. By lemmas 33 and 43, the relation R_2 has a filter $w_1 \prec_2 \ldots \prec_2 w_k$. Now, either R_2 has some filter of range 2 or not. If the first case holds, then by Proposition 54, we can assume that all such filters are either of the form (3.1) or (3.2). Now, it is enough to use Lemma 64. If R_2 does not have any filters of range 2, then the result follows from Lemma 62. □

The following example illustrates Proposition 61.

Example 65. *A relation that does not have a filter of range 2 was presented in Example 63. Here consider $R(x_1, x_2, x_3)$ given by a formula: $(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_3 \leq x_2 \vee x_3 \leq x_1)$. The relation R has two filters minimal wrt $\ll$, namely: $x_3 \prec x_2 \prec x_1$ and $x_1 \approx x_2 \prec x_3$. To show that R is preserved only by essentially unary polymorphisms we will mainly follow the lines of the proof of Lemma 64, which is of leading importance for the proof of Proposition 61.*

First note that $R(v_1, v_1, v_2)$ expresses $(v_1 \leq v_2)$. Define a relation $R_1(x_1, x_2, x_3)$ as $R(x_1, x_2, x_3) \wedge x_3 \leq x_2 \leq x_1$ and observe that R_1 is equivalent to $\neg(x_3 < x_2 < x_1) \wedge x_3 \leq x_2 \leq x_1$. The next step is to show that $R_3(x_1, x_2, x_3)$ defined by $\exists z_1 \exists z_2 R_1(z1, x2, z2) \wedge z1 \leq x1 \leq z2 \wedge z1 \leq x3 \leq z2$ is potentially non-negative. To conclude, note that $R_3 \in [R]$ and use Theorem 20.

3.6 Surjective Unary Polymorphisms of Positive Temporal Relations

This section examines positive temporal relations that are closed only under surjective unary polymorphisms. We want to divide this subset of positive temporal languages into classes each of which contains Γ_1 and Γ_2 if and only if *sPol*(Γ_1) = *sPol*(Γ_2) (or equivalently $[\Gamma_1] = [\Gamma_2]$)). Such a classification facilitate providing complexity results — see Sect. 4.2.

First we give some preliminary definitions. A permutation of a finite set is a bijection from this set to itself. Let $A = \{a_1, \ldots, a_n\}$ be a finite ordered set such that $a_1 < \ldots < a_n$. We say that a permutation π of A is a cycle of A if there exists $i \leq n$ such that $\pi(a_i) < \pi(a_{i+1}) < \ldots < \pi(a_n) < \pi(a_1) < \ldots < \pi(a_{i-1})$. Similarly, π is a reversed cycle if there exists $i \leq n$ such that $\pi(a_i) > \pi(a_{i+1}) > \ldots > \pi(a_n) > \pi(a_1) > \ldots > \pi(a_{i-1})$. Finally π is a transposition if there exists i such that $\pi(a_i) = a_{i+1}$, $\pi(a_{i+1}) = a_i$ and $\pi(a_j) = a_j$ for all $j \notin \{i, i+1\}$. It is well-known that every permutation can be obtained as a composition of transpositions.

Definition 66. *We say that a relation R is closed under all permutations (respectively, under all cycles or reversed cycles) if for every tuple $\langle q_1, \ldots, q_n\rangle \in R$ and every permutation (respectively, every cycle or reversed cycle) π of the set $\{q_1, \ldots, q_n\}$ we have $\langle \pi(q_1), \ldots, \pi(q_n)\rangle \in R$.*

Note that in the definition above we permute the set $\{q_1, \ldots, q_n\}$ (and not the set of indices $\{1, \ldots, n\}$), which may have less then n elements if $q_1, \ldots, q_n$ are not pairwise distinct.

The preceding definitions concerns closure under various kinds of permutations. Although they may look quite similar to closure under polymorphisms, they are different. Below we give some, important for us, examples of (unary) surjective polymorphisms of positive temporal relations. Recall that in Section 3.1 we defined the function $-$ as well as weakly increasing, weakly decreasing, weakly half increasing and weakly half-decreasing surjections on $\mathbb{Q}$. We also said there that a function is weakly monotone if it is either weakly increasing or weakly decreasing; and weakly half-monotone if either weakly half-increasing or weakly half-decreasing.

3.6.1 Interesting Classes of Surjective Unary Polymorphisms

This subsection is devoted to prove the following result.

Theorem 67. *Let Γ be a positive temporal language such that sPol(Γ) contains only essentially unary functions. Then exactly one of the following cases holds.*

1. *The set sPol(Γ) is the set of all unary surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[x_1 = x_2 \vee x_1 = x_3]$. Each relation in Γ is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} = y_{i_1} \vee \ldots \vee x_{i_k} = y_{i_k})$.*
2. *The set sPol(Γ) is the set of all weakly increasing, weakly decreasing or weakly half-monotone surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[R]$ such that R is a conjunction of clauses $(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)} \vee x_{\Pi(3)} \leq x_{\Pi(4)})$ where Π ranges over all cycles and reversed cycles of the set $\{1, 2, 3, 4\}$.*
3. *The set sPol(Γ) is the set of all weakly increasing or weakly decreasing surjections on $\mathbb{Q}$; and then $[\Gamma]$ is equal to $[(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_3 \leq x_2 \vee x_2 \leq x_1)]$.*

4. *The set sPol*(Γ) *is the set of all weakly increasing or weakly half-increasing surjections on* $\mathbb{Q}$*; and then* $[\Gamma]$ *is equal to* $[(x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_2 \leq x_3 \vee x_3 \leq x_1) \wedge (x_3 \leq x_1 \vee x_1 \leq x_2)]$.

5. *The set sPol*(Γ) *is the set of all weakly increasing surjections on* $\mathbb{Q}$*; and then* $[\Gamma]$ *is equal to* $[(x_1 \leq x_2 \vee x_2 \leq x_3)]$.

A similar classification concerning all temporal languages and (not necessarily surjective) unary polymorphisms was obtained in [11]. That classification follows the classical theorem of Cameron [15] that describes temporal constraint languages up to their automorphisms. The authors of [11] showed that all unary non-injective polymorphisms of temporal relations can be locally generated by automorphisms, which in fact means that the classification of unary polymorphisms and automorphisms coincide. Classification of csps allows restricting to injective functions only. (Non-injective functions generate constant functions.) However, in contrast to constraint satisfaction problems, a non-injective surjective polymorphism does not enforce (trivial) tractability in the case of qcsps. Therefore we couldn't restrict ourselves to injective unary surjections. Concluding, although our classification looks similar to the one obtained in [11], it is not clear whether there is a relatively simple way of using Cameron's Theorem in our case.

Weakly half-increasing polymorphisms correspond in some way to the function cyc from [11]. In turn, positive temporal relations preserved by weakly half-decreasing functions correspond to temporal relations closed under the function $-$ and cyc.

As indicated in Theorem 67, there are four interesting classes of unary polymorphisms of positive temporal relations: weakly increasing, weakly decreasing, weakly half-increasing, and weakly half-decreasing. The following lemmas say that if some positive temporal relation is closed under one polymorphism of a given type, then it is closed under all polymorphisms of this type.

Lemma 68. *If sPol*(R) *contains a weakly decreasing unary surjection* f*, then it contains all weakly decreasing unary surjections.*

Proof. Let $g : \mathbb{Q} \to \mathbb{Q}$ be any weakly decreasing surjection. We want to show that R is closed under g. Take any tuple $\langle q_1, \ldots, q_n \rangle \in R$. Let k be the cardinality of $\{q_1, \ldots, q_n\}$. Note that $k \leq n$ and that the cardinality of $\{g(q_1), \ldots, g(q_n)\}$ is bounded by k. Let $a_1, \ldots, a_k$ be any rational numbers such that $a_i \in f^{-1}(i)$ for $i = 1, \ldots, k$. By Lemma 36, the relation R is closed under all strictly increasing functions. Therefore there exists a strictly increasing function $h_0 : \mathbb{Q} \to \mathbb{Q}$ such that $h_0(\{q_1, \ldots, q_n\}) = \{a_1, \ldots, a_k\}$ and a strictly increasing function $h_1 : \mathbb{Q} \to \mathbb{Q}$ such that $h_1(\{1, \ldots, k\}) = \{g(q_1), \ldots, g(q_n)\}$. Then $h_1 f h_0 : \mathbb{Q} \to \mathbb{Q}$ is a surjection that maps $\{q_1, \ldots, q_n\}$ to $\{g(q_1), \ldots, g(q_n)\}$. Moreover, $h_1 f h_0$ is weakly decreasing on $\{q_1, \ldots, q_n\}$, and hence $h_1 f h_0(q_i) = g(q_i)$ for all $i = 1, \ldots, n$. By assumption R is closed under f, so $\langle g(q_1), \ldots, g(q_n) \rangle \in R$ and thus R is closed under g. □

Lemma 69. *Let* R *be a positive temporal relation and let* $f \in$ *sPol*(R)*. If it is possible to choose a strictly increasing sequence* $(a_n)_{n \in \mathbb{N}}$ *such that* $(f(a_n))_{n \in \mathbb{N}}$ *is strictly decreasing, then sPol*(R) *contains the function* $-$.

Proof. The proof follows the lines of the proof of Lemma 68. We just choose h_0 as in Lemma 68 and h_1 to be a strictly increasing function such that $h_1(\{f(a_1), \ldots, f(a_k)\})$ is equal to $\{-(q_1), \ldots, -(q_n)\}$. □

Lemma 70. *If sPol(R) contains a weakly half-increasing unary surjection f, then it contains all weakly half-increasing unary surjections. If sPol(R) contains a weakly half-decreasing unary surjection f, then it contains all weakly decreasing, all weakly half-increasing and all weakly half-decreasing unary surjections.*

Proof. The first implication, similarly as in the proof of Lemma 68, can be proved using the fact that R is closed under all weakly increasing surjections. For the second implication, note that by Lemma 69 the relation R is closed under the function $-$, so by Lemma 68 it is closed under all weakly decreasing surjections. Composing f with the function $-$, we obtain that *sPol*(R) contains some weakly half-increasing surjection, so by previous implication it contains all weakly half-increasing surjections. Finally, by composition of the function $-$ with weakly half-increasing surjections we obtain all weakly half-decreasing surjections. □

Now, we relate various surjective polymorphisms to closures under various kind of permutations (see for example Definition 66).

The next lemma implies that the set of positive temporal relations closed under all permutations equals to the set of positive languages from [5].

Lemma 71. *A positive temporal relation R is closed under all permutations iff sPol(R) contains all essentially unary surjections on $\mathbb{Q}$.*

Proof. Suppose that R is closed under every permutation. Take any unary surjection $f : \mathbb{Q} \to \mathbb{Q}$ and any tuple $\langle q_1, \ldots, q_n \rangle \in R$. Let π be a permutation of $\{q_1, \ldots, q_n\}$ such that $\pi(q_i) \leq \pi(q_j)$ iff $f(q_i) \leq f(q_j)$ for all $i, j \in \{1, \ldots, n\}$. Let $h : \mathbb{Q} \to \mathbb{Q}$ be a strictly increasing function of R such that $h(\pi(q_i)) = f(q_i)$ for all $i \in \{1, \ldots, n\}$. By assumption R is closed under π and of course under each strictly increasing function, which implies that $\langle f(q_1) \ldots, f(q_n)\rangle \in R$.

For the other direction it is enough to consider for any tuple $\langle q_1, \ldots, q_n \rangle \in R$ and any permutation π a unary surjection that restricted to $\{q_1, \ldots, q_n\}$ equals to π. □

Lemma 72. *A positive temporal relation R is closed under all cycles iff sPol(R) contains all weakly half-increasing surjections on $\mathbb{Q}$.*

Proof. Assume that R is closed under all cycles. Take any tuple $\langle q_1, \ldots, q_n \rangle \in R$ and any weakly half-increasing surjection $f : \mathbb{Q} \to \mathbb{Q}$. Let x, y be the two irrational numbers from the definition of f. Let $\{a_1, \ldots, a_k\}$ be the set $\{q_1, \ldots, q_n\}$ with removed duplicates and sorted, that is $\{a_1, \ldots, a_k\} = \{q_1, \ldots, q_n\}$, and $a_1 < a_2 < \ldots < a_k$. If for all $i = 1, \ldots, n$ we have $q_i < x$ or for all i $q_i > x$ then f is weakly increasing on $\{q_1, \ldots, q_n\}$ and $\langle f(q_1), \ldots, f(q_n)\rangle \in R$, so assume that there exists i such that $a_{i-1} < x < a_i$ (note that x is irrational, so both inequalities are strict). Then $f(a_i) < f(a_{i+1}) < \ldots < f(a_n) < y$ and $y < f(a_1) < \ldots < f(a_{i-1})$. By taking an appropriate cycle π and a strictly increasing function h that moves $\pi(a_j)$ to $f(a_j)$ for all j, since R is closed under both of them, we obtain that $\langle f(q_1), \ldots, f(q_n)\rangle \in R$.

The proof for the opposite direction is similar. Assume that R is closed under all weakly half-increasing functions and take tuple $\langle q_1, \ldots, q_n \rangle \in R$ and any cycle π. Let $\{a_1, \ldots, a_k\}$ be as above and let $\pi(a_1) = a_i$. Take any weakly half-increasing function f such that the irrational x in the definition of f satisfies $a_{i-1} < x < a_i$ and a strictly increasing function h that moves $f(a_j)$ to $\pi(a_j)$, we obtain that $\langle \pi(q_1), \ldots, \pi(q_n)\rangle \in R$. □

Lemma 73. *A positive temporal relation R is closed under all reversed cycles iff sPol(R) contains all weakly half-decreasing surjections on $\mathbb{Q}$.*

Proof. If R is closed under all reversed cycles then it is easy to show that it is closed under the function $-$. Then it is closed under all cycles and by Lemma 72 under all weakly half-increasing functions. Composing weakly half-increasing functions with the function $-$ we obtain all weakly half-decreasing functions.

For the opposite direction, if R is closed under all weakly half-decreasing surjections, then by Lemma 69 it is closed under the function $-$. Then it is closed under all weakly half-increasing surjections, so by Lemma 72 it is closed under all cycles. Again using the function $-$ we obtain that R is closed under all reversed cycles. $\square$

Since we are interested in surjective functions, we can claim the following.

Lemma 74. *Let f be an unary, surjective operation on Q, then there exist:*

- *an infinite, strictly monotone sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers such that* $\lim_{n\to\infty} f(a_n) = +\infty$
- *an infinite, strictly monotone sequence $(b_n)_{n\in\mathbb{N}}$ of rational numbers such that* $\lim_{n\to\infty} f(b_n) = -\infty$

Proof. Note that since f is surjective, the sets $f^{-1}(i)$ are not empty for all $i \in \mathbb{N}$. Let s_i be any member of $f^{-1}(i)$. Define $S_i = \{n \in \mathbb{N} \mid s_n \leq s_i\}$ and consider two cases.

- The set $\{j \in \mathbb{N} \mid S_j \text{ is finite}\}$ is infinite.

  Let $j_0 = \min\{j \mid S_j \text{ is finite}\}$ and for all $k \in \mathbb{N}$ define $j_{k+1} = \min\{j \mid S_j \text{ is finite},\ j > j_k,\ j \notin S_{j_k}\}$. Finally put $a_k = s_{j_k}$ for all $k \in \mathbb{N}$. Note that since S_{j_k} is finite, j_{k+1} is well defined. Since $j_{k+1} \notin S_{j_k}$, we have $s_{j_{k+1}} > s_{j_k}$, so $a_{k+1} > a_k$. Since $j_{k+1} > j_k$, we have $f(a_{k+1}) > f(a_k)$ and $f(a_k) \geq k$. Hence $(a_n)_{n\in\mathbb{N}}$ is a strictly monotone (increasing) sequence of rational numbers such that $(f(a_n))_{n\in\mathbb{N}}$ diverges to infinity.

- The set $\{j \in \mathbb{N} \mid S_j \text{ is finite}\}$ is finite.

  Let $j_0 = \min\{j \mid S_j \text{ is infinite}\}$ and for all $k \in \mathbb{N}$ define $j_{k+1} = \min\{j \mid S_j \text{ is infinite},\ j > j_k,\ j \in S_{j_k}\}$. Finally put $a_k = s_{j_k}$ for all $k \in \mathbb{N}$. Note that since S_{j_k} is infinite, j_{k+1} is well defined. Since $j_{k+1} \in S_{j_k}$, we have $s_{j_{k+1}} \leq s_{j_k}$; of course $s_{j_{k+1}} \neq s_{j_k}$, so $a_{k+1} < a_k$. Since $j_{k+1} > j_k$, we have $f(a_{k+1}) > f(a_k)$ and $f(a_k) \geq k$. Hence $(a_n)_{n\in\mathbb{N}}$ is a strictly monotone (decreasing) sequence of rational numbers such that $(f(a_n))_{n\in\mathbb{N}}$ diverges to infinity.

The proof of existence of the sequence $(b_n)_{n\in\mathbb{N}}$ is dual. $\square$

To prove Theorem 67, we show that if *sPol*(Γ) contains any function that is neither weakly monotone nor weakly half-monotone, then it contains all unary rational functions or equivalently, by Lemma 71, is closed under all permutations.

Lemma 75. *Let R be a positive relation. If there exists $f \in$ sPol(R) that satisfies one of the following conditions then R is closed under all permutations. Conditions are as follows:*

1. *For every $k \in N$ there exists a sequence of rational numbers*

   (a) $a < a_1 < \ldots < a_k < b_1 < \ldots < b_k$ *or*

   (b) $a_1 < \ldots < a_k < b_1 < \ldots < b_k < a$

   such that $f(a_1) < \ldots < f(a_k) < f(a) < f(b_1) < \ldots < f(b_k)$.

2. *For every $k \in N$ there exists a sequence of rational numbers $a_0 < \ldots < a_k < b_1 < \ldots < b_k$ such that $f(a_k) < \ldots < f(a_0) < f(b_1) < \ldots < f(b_k)$.*

3. *For every $k \in N$ there exists a sequence of rational numbers $a_1 < \ldots < a_k < b_0 < \ldots < b_k$ such that $f(a_1) < \ldots < f(a_k) < f(b_k) < \ldots < f(b_0)$.*

4. *For every $k \in N$ there exists a sequence of rational numbers $a_1 < \ldots < a_k < a < b_1 < \ldots < b_k$ such that $f(b_1) < \ldots < f(b_k) < f(a) < f(a_1) < \ldots < f(a_k)$.*

5. *For every $k \in N$ there exists a sequence of rational numbers $a_1 < \ldots < a_k < b_1 < \ldots < b_k$ such that $f(a_1) < \ldots < f(a_{k-1}) < f(b_1) < f(a_k) < f(b_2) < \ldots < f(b_k)$.*

Proof. Take any tuple $\langle q_1, \ldots, q_n \rangle \in R$. Let $\{c_0, \ldots, c_k\}$ be the set $\{q_1, \ldots, q_n\}$ with removed duplicates and sorted, that is $\{c_0, \ldots, c_k\} = \{q_1, \ldots, q_n\}$, and $c_0 < c_1 < \ldots < c_k$. For each of the conditions 1–5, it is enough to show that R is closed under every transposition of adjacent elements in $\{c_0, \ldots, c_k\}$.

Case 1a Let $a, a_1 \ldots, a_k, b_1, \ldots, b_k$ be as in Condition 1a.

First we show that R is closed under all cycles. Let $h_0 : \mathbb{Q} \to \mathbb{Q}$ be a strictly increasing function such that $h_0(\{c_0, \ldots, c_k\}) = \{a, a_1 \ldots, a_k\}$. Let h_1 be a strictly increasing function such that $h_1(\{f(a), f(a_1), \ldots, f(a_k)\}) = \{c_0, \ldots, c_k\}$ and let h be $h_1 f h_0$. Since h_0 and h_1 are strictly increasing functions, R is closed also under h and we have $\langle h(q_1), \ldots, h(q_n) \rangle \in R$. Observe that h is a cycle of $\{c_0, \ldots, c_k\}$ such that $h(c_1) < h(c_2) < \ldots < h(c_k) < h(c_0)$. Composing h with itself we get that R is closed under all cycles.

Now we show that R is closed under the transposition that maps c_0 to c_1 and c_1 to c_0. Let h_2 be a strictly increasing function such that $h_2(c_0) = a$, $h_2(c_1) = a_1$ and $h_2(c_j) = b_j$ for $j = 2, \ldots, k$, h_3 such that $h_1(\{f(a), f(a_1), f(b_3) \ldots, f(b_k)\}) = \{c_0, \ldots, c_k\}$, and let h' be $h_3 f h_2$. Again R is closed under h', and h' is the desired transposition of $\{c_0, \ldots, c_k\}$.

Since R is closed under all cycles and under a transposition, it is also closed under all transpositions and thus under all permutations.

Observe that in the above reasoning we used elements $a, a_1, \ldots, a_k$ to obtain all cycles and elements $a, a_1, b_2, \ldots, b_k$ to obtain a transposition. In the following, we do not go into details of functions h_j, we only say which elements are used to obtain a particular permutation.

Case 1b Here, using elements $b_1, \ldots, b_k, a$ we can obtain any cycle and using $a_1, \ldots, a_{k-1}, b_1, a$ we obtain a transposition that swaps two greatest elements.

Case 2 Using elements $a_0, \ldots, a_i, b_{i+1}, \ldots, b_k$ we obtain a permutation π_i that reverses the order of the smallest $i + 1$ elements, that is $\pi_i(c_i) < \pi_i(c_{i-1}) < \ldots < \pi_i(c_0) < \pi_i(c_{i+1}) < \pi_i(c_{i+2}) < \ldots < \pi_i(c_k)$. Then $\pi_i \pi_1 \pi_i$ is a transposition that swaps c_i with c_{i+1}.

Case 3 This case is symmetric to the previous one. Using elements $b_0, \ldots, b_i$, $a_{i+1}, \ldots, a_k$ we obtain a permutation π'_i that reverses the order of the greatest $i + 1$ elements in $\{c_0, \ldots, c_k\}$. Then $\pi'_i \pi'_1 \pi'_i$ is a transposition that swaps c_{k-i} with c_{k-i+1}.

Case 4 Here, using elements $a_1, \ldots, a_k, a$ we can obtain any cycle. Using elements $a_1, \ldots, a_{k-1}, a, b_1$ we obtain a permutation π such that $\pi(c_k) < \pi(c_{k-1}) < \pi(c_0) < \pi(c_1) < \ldots < \pi(c_{k-2})$. Composing it with a cycle that moves two first elements to the end, we get a transposition that swaps two greatest elements. Again composing this transposition with cycles we get arbitrary transpositions and thus arbitrary permutations.

Case 5 Using elements $a_0, \ldots, a_{i-1}, a_k, b_1, b_{i+2}, \ldots, b_k$ we get a transposition that swaps c_i with c_{i+1}. □

The general idea we follow (in Lemma 77) is that if a considered surjection is neither monotone nor half-monotone, then we can choose appropriate subsequences of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ from Lemma 74 to satisfy some condition in Lemma 75. Lemma 76 handles one hard special case in the proof of Lemma 77.

Lemma 76. *Let R be a positive temporal relation such that sPol(R) contains a function f that is not weakly half-increasing. Let $(c_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence and let $(d_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence such that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = x \notin \mathbb{Q}$ and $\lim_{n\to\infty} f(c_n) = +\infty$ and $\lim_{n\to\infty} f(d_n) = -\infty$. Then R is closed under all permutations.*

Proof. Note that in the surrounding of x the function f is not monotone, but it still could be increasing on both sets $\{q \in \mathbb{Q} \mid q < x\}$ and $\{q \in \mathbb{Q} \mid q > x\}$. Since f is not weakly half-increasing, we have to consider three following cases.

- f restricted to the set $\{q \in \mathbb{Q} \mid q < x\}$ is not weakly increasing.

  In this case there exist $a_0 < x$ and $a_1 < x$ such that $a_0 < a_1$ and $f(a_0) > f(a_1)$. Let n_0 be big enough so that $f(c_{n_0}) > f(a_0)$ and $c_{n_0} > a_0$ and define for $i = 2, \ldots, n$, where n is the arity of R, $a_i = c_{n_0+i}$. Let m_0 be big enough so that $f(d_{m_0-n}) < f(a_1)$ and define $b_i = d_{m_0-n+i}$. Now using elements $a_1, \ldots, a_n, b_1, \ldots b_n$ we can show (in a way very similar to Lemma 75) that R is closed under all cycles; using elements $a_0, a_1, \ldots, a_n$ we can show that it is closed under a transposition that swaps two smallest elements. Hence it is closed under all permutations.

- f restricted to the set $\{q \in \mathbb{Q} \mid q > x\}$ is not weakly increasing.

  This case is very similar to the previous one. The difference is that using non-monotonicity of the other half of f we can generate a transposition that swaps two greatest elements in a sequence.

- for all irrational y there exists $q \in \mathbb{Q}$ such that $q < x$ and $f(q) < y$, or $q > x$ and $f(q) > y$.

  It is easy to show that in this case there exists a real number y_0 and rational numbers a_0, b_0 such that $a_0 < x < b_0$ and $f(a_0) < y_0 < f(b_0)$. Let n_0 be such that $c_{n_0} > a_0$ and $f(c_{n_0}) > f(b_0)$. Let m_0 be such that $d_{m_0-n} < b_0$ and $f(d_{m_0-n}) < f(a_0)$. For $i = 1, \ldots, n$ define $a_i = c_{n_0+i}$ and $b_i = d_{m_0-n+i}$. Now using elements $a_1, \ldots, a_n, b_1, \ldots b_n$ we can show that R is closed under all cycles; using elements $a_0, a_1, \ldots, a_k, b_0$ and the fact that $f(a_0) < f(b_0) < f(a_1) < \ldots < f(a_k)$ we show that R is closed under a permutation π that maps the greatest elements to the second position: $\pi(a_1) < \pi(a_k) < \pi(a_2) < \ldots < \pi(a_{k-1})$. Composing this permutation with the cycle π' such that $\pi'(a_2) = a_1$ and $\pi'(a_2) < \ldots < \pi'(a_k) < \pi'(a_1)$ we obtain $\pi(\pi'(a_2)) < \pi(\pi'(a_1)) < \pi(\pi'(a_3)) < \ldots < \pi(\pi'(a_k))$, so $\pi\pi'$ is a transposition. Hence R is closed under all cycles and under a transposition, so it is closed under all permutations.

  □

Lemma 77. *Let R be a positive temporal relation such that sPol(R) contains a unary surjection f that is neither weakly motone nor weakly half-monotone. Let $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ be two strictly monotone sequences such that $\lim_{n\to\infty} f(c_n) = +\infty$ and $\lim_{n\to\infty} f(d_n) = -\infty$. Then R is closed under all permutations.*

Proof. Let $c = \lim_{n\to\infty} c_n$ and $d = \lim_{n\to\infty} d_n$ where both these limits are meant in the set of real numbers with $+\infty$ and $-\infty$. We will consider 8 cases, depending on whether $(c_n)_{n\in\mathbb{N}}$ is increasing (we will write $c\nearrow$) or decreasing ($c\nwarrow$), whether $(d_n)_{n\in\mathbb{N}}$ is increasing or decreasing ($d\searrow$ or $d\swarrow$, respectively) and whether $c \le d$ or $c > d$.

Case $c\nearrow, d\searrow, c \ge d$. Take any $k \in \mathbb{N}$. Let n_0 be such that $f(d_{n_0}) < 0$; let m_0 be such that $f(c_{m_0}) > 0$ and $c_{m_0} > d_{n_0+k}$. For $i = 0, \dots, k$ define $a_i = d_{n_0+i}$ and $b_i = c_{m_0+i}$. Then the assumptions of Lemma 75, condition 2 are satisfied and thus R is closed under all permutations.

Case $c\nearrow, d\searrow, c \le d$. This case is dual to the previous one in the following sense. Since $(d_n)_{n\in\mathbb{N}}$ is increasing, by Lemma 69 the relation R is closed under the function $-$, so it is closed under $-(f)$. Then $-(f)$ is a function as in the previous case, with $\lim_{n\to\infty}(-f(c_n)) = -\infty$ and $\lim_{n\to\infty}(-f(d_n)) = +\infty$. Hence, by previous case, R is closed under all permutations.

Case $c\nwarrow, d\swarrow, c \ge d$. Take any $k \in \mathbb{N}$. Let n_0 be such that $f(c_{n_0}) > 0$; let m_0 be such that $f(d_{m_0}) < 0$ and $d_{m_0} < c_{n_0+k}$. For $i = 1, \dots, k$ define $a_i = d_{m_0+k-i}$ and for $i = 0, \dots, k$ define $b_i = c_{n_0+k-i}$. Then the assumptions of Lemma 75, condition 3 are satisfied and thus R is closed under all permutations.

Case $c\nwarrow, d\swarrow, c \le d$ This case is dual to the previous one.

Case $c\nearrow, d\swarrow, d < c$ Since f is not increasing, there exists $a < b$ such that $f(a) > f(b)$. Consider three cases

- $d < a < b < c$

  Take any $k \in \mathbb{N}$. Let n_0 be such that $f(c_{n_0}) > f(a)$ and $c_{n_0} > b$, let m_0 be such that $f(d_{m_0-k}) < f(b)$ and $d_{m_0-k} < a$. Define $a_i = d_{m_0-i}$ for $i = 1, \dots, k-1$ and $a_k = a$. Define $b_1 = b$ and $b_i = c_{n_0+i}$ for $i = 2, \dots, k$. Then the assumptions of Lemma 75, condition 5 are satisfied and thus R is closed under all permutations.

- $a \le d$

  This case reduces to Lemma 75, condition 1a.

- $b \ge c$

  This case reduces to Lemma 75, condition 1b, where b plays the role of a from the formulation of the lemma.

Case $c\nwarrow, d\searrow, c < d$ This case is dual to the previous one.

Case $c\nearrow, d\swarrow, c \le d$ If there exists a rational number a such that $c \le a \le d$, then this case reduces to Lemma 75, condition 4. Otherwise it is handled by Lemma 76.

Case $c\nwarrow, d\searrow, d \le c$ This case is dual to the previous one. □

Proof of Theorem 67 (Part One) Suppose that the set *sPol*(Γ) contains only essentially unary surjections. If *sPol*(Γ) contains a function f that is neither weakly monotone nor weakly half-monotone, then by Lemma 74 we find two strictly monotone sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f(a_n) = +\infty$ and $\lim_{n\to\infty} f(b_n) = -\infty$. Then by Lemma 77 every relation in Γ is closed under all permutations, so by Lemma 71 *sPol*(R) contains all essentially unary surjections. Hence *sPol*(Γ) is the set of all essentially unary surjections on $\mathbb{Q}$ and we are in case 1.

Now assume that *sPol*(Γ) contains only weakly monotone or weakly half-monotone surjections. There are four cases, depending on whether *sPol*(Γ) contains a weakly decreasing surjection or a weakly half-increasing surjection.

If *sPol*(Γ) contains a weakly decreasing surjection and a weakly half-increasing surjection, then, since it is closed under composition of unary surjections, it contains a weakly half-decreasing surjection and by Lemmas 68 and 70, it contains all weakly decreasing and all weakly half-monotone surjections on $\mathbb{Q}$, so we are in case 2.

If *sPol*(Γ) contains a weakly decreasing surjection and it does not contain any weakly half-increasing surjection, then by lemmas 68 and 70 it contains all weakly decreasing and it does not contain any weakly half-monotone surjections, so we are in case 3.

If *sPol*(Γ) does not contain any weakly decreasing surjection and it contains a weakly half-increasing surjection, then by lemmas 68 and 70 it contains all weakly half-increasing and it does not contain weakly decreasing surjections, so we are in case 4.

Finally, if *sPol*(Γ) does not contain any weakly decreasing surjection and it does not contain any weakly half-increasing surjection, then by lemmas 68 and 70 it does not contain any weakly decreasing nor weakly half-monotone surjection, so we are in case 5.

(PartTwo) Recall from [5] that $(x_1 = x_2 \vee x_1 = x_3)$ as well as all relations definable as $\bigwedge_{i=1}^{n}(x_{i_1} = y_{i_1} \vee \ldots \vee x_{i_k} = y_{i_k})$ is closed under all essentially unary surjections on $\mathbb{Q}$ — see also Proposition 22 in Section 2.3. Now, for each of the classes 2–5 of Theorem 67 we show that a relation given by the formulation of theorem represents the qc-definability of each of these classes. By Theorem 20, it is enough to check whether presented relations are closed under appropriate polymorphisms. We denote these relations $R_{(2)}$–$R_{(5)}$ where $R_{(i)}$ concerns the i-th item of the theorem.

The relation $R_{(5)}$ defined by $R_{(5)}(x_1, x_2, x_3) := (x_1 \leq x_2 \vee x_2 \leq x_3)$, as all positive temporal relations, is closed under all weakly increasing functions. Observe that $\langle 1, 2, 3\rangle \in R_{(5)}$, but $\langle -1, -2, -3\rangle \notin R_{(5)}$, so $R_{(5)}$ is not closed under weakly decreasing functions (and by Lemma 70 it is not closed under half-decreasing functions). Similarly, $\langle 1, 3, 2\rangle \in R_{(5)}$, but $\langle 3, 2, 1\rangle \notin R_{(5)}$, so $R_{(5)}$ is not closed under cycles (and thus it is not closed under weakly half-increasing functions).

The relation $R_{(4)}$ defined by $R_{(4)}(x_1, x_2, x_3) := (x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_2 \leq x_3 \vee x_3 \leq x_1) \wedge (x_3 \leq x_1 \vee x_1 \leq x_2)$ is a conjunction of the relations $(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)})$ where Π ranges over all cycles of the set $\{1, 2, 3\}$, so it is closed under all cycles. Since $\langle 1, 2, 3\rangle \in R_{(4)}$ and $\langle 3, 2, 1\rangle \notin R_{(4)}$, it is not closed under weakly decreasing or weakly half-decreasing functions.

It is easy to observe that the relation $R_{(3)}$ defined by $R_{(3)}(x_1, x_2, x_3) := (x_1 \leq x_2 \vee x_2 \leq x_3) \wedge (x_3 \leq x_2 \vee x_2 \leq x_1)$ is closed under weakly decreasing functions. Since $\langle 2, 1, 3\rangle \in R_{(3)}$ and $\langle 3, 2, 1\rangle \notin R_{(3)}$, this relation is not closed under cycles and by Lemmas 72 and 70 it is not closed under any weakly half-monotone surjection.

Let a relation $R_{(2)}$ be defined as a conjunction of the clauses $(x_{\Pi(1)} \leq x_{\Pi(2)} \vee x_{\Pi(2)} \leq x_{\Pi(3)} \vee x_{\Pi(3)} \leq x_{\Pi(4)})$ where Π ranges over all cycles and reversed cycles of the set $\{1, 2, 3, 4\}$. Thus the relation $R_{(2)}$ must obviously be closed under all cycles and reversed cycles. Note that cycles and reversed cycles are 8 out of total 24 permutations of the set $\{1, 2, 3, 4\}$ (this explains why we could not use a ternary relation as the representative here — all permutations of the set $\{1, 2, 3\}$ are either cycles or reversed cycles). To see that $R_{(2)}$ is not closed under all permutations observe that $\langle 4, 3, 2, 1\rangle \notin$

$R_{(2)}$, but $\langle 2, 1, 3, 4\rangle \in R_{(2)}$.

Finally, we will show that all these relations ($R_{(2)}$–$R_{(5)}$) are closed under essentially unary surjections only. It is done in the same way as in Example 63. Let $R(x_1, \ldots, x_k)$ where $k = 3, 4$ be one of these four relations. Then a relation given by $\bigwedge_{\Pi \in S_k} R(x_{\Pi(1)}, \ldots, x_{\Pi(k)})$ where S_k is a set of all permutations on k elements is equivalent to a relation R' defined by $\bigvee_{i \neq j} x_i = x_j$. Because R' is positive and non-negative, by Corollary 25 and Theorem 20, we have that R is closed under unary surjections only. $\square$

Chapter 4

Complexity Proofs

In chapter 3 we partitioned positive temporal languages into nine classes. This one is devoted to arm each of these families with the complexity proof of the connected qcsp. Some of these computational problems were considered before and hence we know the complexity for families 1, 2 and 5 of Theorem 29. These results are given in the formulation of Theorem 6. Here we complete this picture by giving the rest of complexity proofs. Subsection 4.1 handles families 3 and 4 whereas subsection 4.2 takes care of items 6-9, which corresponds to languages closed under some non-trivial subset of unary surjections on $\mathbb{Q}$.

The other thing we consider in this chapter is the meta-problem of qcsps for positive temporal relations. Finally, we give the proof of our main theorem — Theorem 6.

4.1 PTIME-complete Positive Temporal Languages

Here, we give the complexity of *QCSP* for languages whose relations may be defined as a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$; but they can be neither defined as a conjunction of equalities nor as a conjunction of inequalities. By Theorem 29, for each such language Γ we have $[\Gamma] = [\Gamma_{Left}]$ and hence it is enough to examine the complexity of *QCSP*(Γ_{Left}). In this section, if it is not stated otherwise, whenever we write: a formula, we think of a qc-formula over Γ_{Left}. Recall from Section 3.4 that Γ_{Left} is a set of relations R^k_{Left} for each natural $k \geq 2$. Observe also that the definition of each R^k_{Left} has a simple tree-like structure where x_1 is a *root* and $x_2, \ldots, x_k$ are *sons* of x_1. Moreover, to denote a root of a clause C we write $root(C)$ and to denote a set of sons – $sons(C)$.

The reasoning we give in the following may be easily adopted to languages of relations definable by formulas of the form $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$.

Lemma 78. *Let ϕ be a qc-formula defining a positive temporal relation. Assume that ϕ contains clauses $C_1 := (y \leq x_1 \vee \ldots \vee y \leq x_k)$ and $C_2 := (x_1 \leq z_1 \vee \ldots \vee x_1 \leq z_l)$. Then ϕ and ϕ' given by $\phi \wedge C_3$, where $C_3 := (y \leq z_1 \vee \ldots \vee y \leq z_l \vee y \leq x_2 \vee \ldots \vee y \leq x_k)$, are equivalent, that is, they define the same relation.*

Proof. It is clear that if some valuation of variables satisfies ϕ', then it also satisfies ϕ. Now, we show the converse. In fact, we claim that C_3 is satisfied if C_1 and C_2 are. Let q be a valuation of ϕ. If $q(y) \leq q(x_1)$ and $q(x_1) \leq q(z_j)$ for some $1 \leq j \leq l$, then

$q(y) \leq q(z_j)$ and therefore C_3 is satisfied. Otherwise, we have $q(y) \leq q(x_i)$ for some $2 \leq i \leq k$. Hence, in this case C_3 is satisfied as well. □

We sometimes refer to a quantifier-free formula ϕ as to a set of clauses. In a similar way, we refer to a clause as a set of disjuncts (inequalities). We say that a set of clauses ϕ is $TClosed$ if for all pairs of clauses of the form $(y \leq x_1 \vee \ldots \vee y \leq x_k)$ and $(x_1 \leq z_1 \vee \ldots \vee x_1 \leq z_l)$, the clause $(y \leq z_1 \vee \ldots \vee y \leq z_l \vee y \leq x_2 \vee \ldots y \vee x_k)$ also belongs to ϕ. By $TClosure(\phi)$ we denote the least $TClosed$ superset of ϕ.

By a simple induction, from Lemma 78 we can obtain the following.

Corollary 79. *Formulas ϕ and $TClosure(\phi)$ are equivalent.*

In this section, we see a sentence ψ as a two-player game — we refer the reader to Section 2.2. We say that an inequality $x \leq y$ is completely evaluated at some moment during the game if both x and y are already evaluated.

We show that the universal player has a winning strategy if and only if $TClosure(\phi)$ contains a clause of the form $(y \leq x_1 \vee \ldots \vee y \leq x_k)$ such that for each disjunct $y \leq x_i$ where $1 \leq i \leq k$ we have that the later of the two variables y and x_i is universal. We call such a clause *ultimate*.

Lemma 80. *Let ψ be a qc-formula without free variables with a quantifier prefix Q and a quantifier-free part ϕ. Then ψ is false if and only if $TClosure(\phi)$ contains an ultimate clause.*

Proof. If $TClosure(\phi)$ contains an ultimate clause, then the universal player is able to falsify each its inequality and hence, by Lemma 78, to falsify ψ.

For the converse we show the following: if $TClosure(\phi)$ does not contain an ultimate clause, then the existential player has a winning strategy.

First, we describe a strategy and then we show that it is indeed winning. Consider some moment of the game that is just before some move of the existential player. Suppose that she is going to evaluate some variable x. Due to our strategy she considers two sets of clauses. By $CriticalRoot(x)$ we denote a set of clauses C such that $root(C) = x$ and all other variables are either already evaluated or universal. The second of interesting sets is denoted by $CriticalSon(x)$. This set consists of clauses C such that $x \in sons(C)$ and all existential variables except for x as well as $root(C)$ are already evaluated in the way that all completely evaluated disjuncts (inequalities) are false.

She computes the value $\mathbf{sup}$ equal to $\min_{C \in CriticalRoot(x)} \max_{(x \leq y) \in C} q(y)$ where we take into account only variables y that are already evaluated. If $CriticalRoot(x)$ is empty, then we evaluate $\mathbf{sup}$ to $+\infty$. Similarly, the existential player computes $\mathbf{inf}$ equal to $\max_{C \in CriticalSon(x)} q(root(C))$. If $CriticalSon(x)$ is empty, then we set $\mathbf{inf}$ to $-\infty$. Finally, the existential player evaluates x to some number from the interval $\langle \mathbf{inf}, \mathbf{sup} \rangle$. If $\mathbf{inf} > \mathbf{sup}$, then x is evaluated arbitrarily.

We now prove that the strategy given above is in fact winning.

Assume the contrary. Then there exists a game such that the existential player looses playing this strategy. If the universal player wins, then there exists a move and a clause C such that after this move all completely evaluated disjuncts of C are false and not yet evaluated variables are universal. We call this move critical. Consider the first critical move during the game. Because there are no ultimate clauses in $TClosure(\phi)$, the critical move is existential. For the same reason, there exists some critical move, that is, we can assume that the game is not lost before the first move. Let x be the variable evaluated in this move. Further, before this move holds $\mathbf{inf} > \mathbf{sup}$. Therefore there are two clauses: C_1 defined by $(y \leq x \vee y \leq x_1 \vee \ldots \vee y \leq x_k)$ and C_2 equal to $(x \leq z_1 \vee \ldots \vee x \leq z_l)$ such that $C_1 \in \mathit{CriticalSon}(x)$, $C_2 \in \mathit{CriticalRoot}(x)$; and y is strictly greater than each already evaluated x_i for $1 \leq i \leq k$ and each already evaluated z_j for $1 \leq j \leq l$.

By Lemma 78, the set $TClosure(\phi)$ contains a clause C_3 equal to $y \leq z_1 \vee \ldots \vee y \leq z_l \vee y \leq x_1 \vee \ldots \vee y \leq x_k$. Observe that all not yet evaluated variables are universal and all evaluated disjuncts are false. Therefore there is some earlier critical move during this game, which contradicts the assumption that we consider the first one. □

To show the exact complexity of *QCSP*(Γ_{Left}) we use the emptiness problem for context-free grammars. It is well known that this problem is P-complete. We assume that the reader is familiar with the notion of the context-free grammar [27]. By $\mathcal{L}(G)$ we denote a language generated by a context free-grammar $G = \langle N, \Sigma, R, S \rangle$, where N is the set of nonterminals, Σ is the set of terminals, R is the set of productions, and S is the starting symbol of G.

Theorem 81. *Let Γ be a positive temporal language such that each of its relation is definable by a formula of the form $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ and it is neither definable as a conjunction of equalities nor as a conjunction of inequalities. Then the problem QCSP(Γ) is P-complete*

Proof. By Theorem 44 and Lemma 47 it is enough to examine the complexity of the problem *QCSP*$(\{x_1 \leq x_2 \vee x_1 \leq x_3\})$.

(About Membership) To obtain the result we give a logspace reduction from the problem *QCSP*$(\{x_1 \leq x_2 \vee x_1 \leq x_3\})$ to the emptiness problem for context-free grammars. Let ψ be an instance of *QCSP*$(\{x_1 \leq x_2 \vee x_1 \leq x_3\})$ with a quantifier prefix Q and a quantifier-free part ϕ. We construct a context-free grammar G_ψ such that $\mathcal{L}(G_\psi) \neq \emptyset$ if and only if ψ is false. By Lemma 80, it is enough to construct G_ψ that generates a non-empty language if and only if $TClosure(\phi)$ contains an ultimate clause.

The reduction runs as follows. For each variable x of ψ, we show a grammar G^x_ψ that generates an empty language if and only if there is no ultimate clause C in $TClosure(\phi)$ with x being a root of C. Further, we define G_ψ as a grammar such that $\mathcal{L}(G_\psi) = \bigcup_{x \in \mathit{Var}(\psi)} \mathcal{L}(G^x_\psi)$ where $\mathit{Var}(\psi)$ is a set of all variables of ψ. Recall that the set of context-free grammars is closed under union and note that $\|G_\psi\| \leq c * \Sigma_{x \in \mathit{Var}(\psi)} \left\| G^x_\psi \right\|$ for some constant c.

We now turn to the definition of $G_x = \langle N_x, \Sigma_x, R_x, A_x \rangle$. For each variable y of ψ we introduce a nonterminal A_y. The set Σ_x contains a terminal symbol a_y for each variable y that is universal and later than x. If x is universal, then there is also a terminal a_y for each variable y that is earlier than x. Further, for each clause of the form $(x_1 \leq x_2 \vee x_1 \leq x_3)$ we have a rule $A_{x_1} \rightarrow A_{x_2} A_{x_3}$. For each terminal symbol a_y in Σ_x there is also a rule $A_y \rightarrow a_y$. It is clear, that such a reduction may be provided using logarithmic space.

Now, if $\mathcal{L}(G^x_\psi)$ contains a word $a_{x_1} \ldots a_{x_k}$, then, by a simple induction, we can show that a clause $(x \leq x_1 \vee \ldots \vee x \leq x_k)$ belongs to $TClosure(\phi)$. Since each x_i for $1 \leq i \leq k$ is universal and later than x or provided x is universal, earlier than x; this clause is ultimate. Similarly, if any ultimate clause $(x \leq x_1 \vee \ldots \vee x \leq x_k)$ belongs to $TClosure(\phi)$, then we can construct a parse tree that witnesses $a_{x_1} \ldots a_{x_k} \in \mathcal{L}(G^x_\psi)$

(About Hardness) We give a logspace reduction from the emptiness problem for context-free grammars to the problem *QCSP*$(\{x_1 \leq x_2 \vee x_1 \leq x_3\})$. We start with a context-free grammar $G = \langle N, \Sigma, S, R \rangle$ in the Chomsky normal form. Then we build a qc-formula ψ_G with a quantifier prefix Q_G and a quantifier free part ϕ_G over $(x_1 \leq x_2 \vee x_1 \leq x_3)$ such that $\mathcal{L}(G)$ is empty if and only if $TClosure(\phi_G)$ contains no ultimate clause.

We have two types of variables. For each nonterminal $A \in N$ we introduce a variable x_A and for each terminal symbol a there is a variable x_a. Variables that come from nonterminals are all existential and those that comes from terminals are universal. Concerning a quantifier prefix Q_G we order variables in the following manner. The variable x_S is the first, then come variables that originate from terminals. Finally, at the end of Q_G we put variables that come from $N \setminus \{S\}$. In the quantifier free part we have a clause $(x_A \leq x_B \vee x_A \leq x_C)$ for each rule of the form $A \to BC$ and a clause $(x_A \leq x_a)$ for each production $A \to a$. The reduction may be of course provided using logarithmic space.

We now claim that $\mathcal{L}(G)$ is not empty if and only if $TClosure(\phi_G)$ contains some ultimate clause. If $\mathcal{L}(G)$ contains a word $a_1 \ldots a_n \in \Sigma^*$ then by a simple induction we can show that the clause C of the form $(x_S \leq x_{a_1} \vee \ldots \vee x_S \leq x_{a_n})$ belongs to $TClosure(\phi_G)$. Because each x_{a_i} for $1 \leq i \leq n$ is universal and later than x_S, the clause C is ultimate. Similarly, if there is some ultimate clause C in $TClosure(\phi)$, then it must be of the form $(x_S \leq x_{a_1} \vee \ldots \vee x_S \leq x_{a_n})$. Now, it is not hard to show that $a_1 \ldots a_n \in \mathcal{L}(G)$. □

4.2 PSPACE-complete Positive Temporal Languages

Recall from Section 1.4 the complexity characterization of equality languages. By corollaries 25 and 21, the problem *QCSP*(Γ) where Γ is closed under essentially unary functions only is NP-hard. Likewise, we know that *QCSP* for languages from item 5 of Theorem 29 is NP-complete. This section is devoted to show PSPACE-completness for qcsps of languages from items 6–9 of Theorem 29.

Membership in PSPACE is the simpler part of the proof and is common for all, not only positive, temporal relations.

Proposition 82. *For every temporal language Γ, the problem QCSP(Γ) is decidable in PSPACE.*

Proof. The set *sPol*(Γ) for each temporal language Γ contains all functions that preserve order i.e., strictly increasing functions. Such functions are also automorphisms of Γ. Therefore each positive temporal relation $R(x_1, \ldots, x_n)$ can be represented as a set of weak linear orders over the set $\{1, \ldots, n\}$.

To solve *QCSP*(Γ) for some positive temporal language Γ we can use a standard backtracking algorithm. It branches on the outermost quantifier and sets an appropriate variable to some value from the set $\{1, \ldots, n\}$. Because there are only n choices here, the algorithm works in polynomial space. □

In the rest of the section we prove hardness. Note that the set of surjective polymorphisms from item 2 contains sets of surjective polymorphisms from each of items 2–5. Therefore, by Theorem 20 and Corollary 21, it is enough to show PSPACE-hardness of *QCSP* for positive temporal languages closed under all weakly monotone and all weakly half-montone surjections only.

Theorem 83. *Let Γ be a set of positive languages closed only under essentially unary functions. If sPol(Γ) is the set of all weakly increasing, weakly decreasing and weakly half-monotone surjections of $\mathbb{Q}$, then QCSP(Γ) is PSPACE-hard.*

Because of Theorem 20 and Corollary 21, it is enough to choose just one language with appropriate set of surjective polymorphisms and show PSPACE hardness for this language. Our choice is the language Γ_{Circle} defined below. We show that it is closed only under all weakly increasing, weakly decreasing and weakly half-monotone surjections of $\mathbb{Q}$. In fact, it is enough to show that Γ_{circle} is closed only under unary surjections of $\mathbb{Q}$ and that is closed under all reversed cycles – see lemmas 70 and 73. Finally, we show that *QCSP*(Γ_{Circle}) is PSPACE-hard and in consequence we prove Theorem 83.

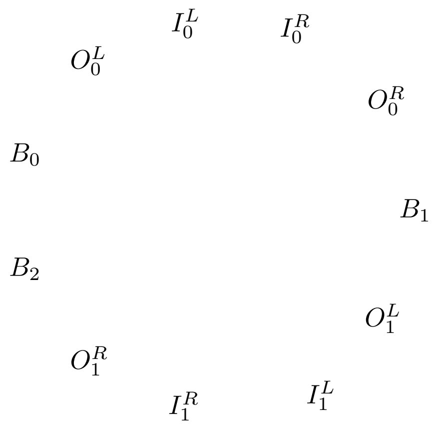

Figure 4.1: The representation of the set $Arenas$

4.2.1 Definition of Γ_{Circle}

First we present some auxiliary relations that shorten the definition. We will use the set of variables $Var_A = \{B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2\}$ whose elements are called *arena variables*. They range over $\mathbb{Q}$. Maybe, it will be easier for the reader to remember the names of arena variables if we explain them. The letter O stands for 'outside' and the letter I for 'inside'. In some sense, which is explained below, 'inside' variables are preffered to be between 'outside' variables with the same lower index. In turn, letters L and R stands for 'left' and 'right'. Finally, the letter B stands for 'border' and variables with this letter in its name have to separate 'inside' and 'outside' variables with different lower indexes.

Let us consider vectors of arena variables. Each of these vectors may be seen as a function from the set $\{0, \ldots, 10\}$ to the set of arena variables. Therefore we feel free to say that one of these vectors is a permutation of another one. Let $\vec{v_A} = \langle B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2\rangle$. We restrict our attention to cycles and reversed cycles of this vector, that is, to vectors listed on Fig. 4.2. We call the set of these vectors $Arenas$.

Observe that the set $Arenas$ may be represented in some way using Fig. 4.1. To obtain one of vectors that is represented by this circle, we have just to tear it apart and orientate. If we orientate it clockwise, then we represent some cycle of $\vec{Var}_A$, that is, one of vectors 1–11 from Fig. 4.2. Otherwise, if we orientate it anticlockwise, then we represent some reversed cycle of this vector, that is, one of vectors 12–22.

Now, for each $\vec{v} \in Arenas$ we define a relation $Prefix_{\vec{v}} := y_0 \geq y_1 \vee \ldots \vee y_9 \geq y_{10}$ where $\vec{v} = \langle y_0, \ldots, y_{10}\rangle$. Let f be an appropriate function from $\{0, \ldots, 10\}$ to

1. $\langle B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2 \rangle$
2. $\langle O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2, B_0 \rangle$
3. $\langle I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2, B_0, O_0^L \rangle$
4. $\langle I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2, B_0, O_0^L, I_0^L \rangle$
5. $\langle O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2, B_0, O_0^L, I_0^L, I_0^R \rangle$
6. $\langle B_1, O_1^L, I_1^L, I_1^R, O_1^R, B_2, B_0, O_0^L, I_0^L, I_0^R, O_0^R \rangle$
7. $\langle O_1^L, I_1^L, I_1^R, O_1^R, B_2, B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1 \rangle$
8. $\langle I_1^L, I_1^R, O_1^R, B_2, B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L \rangle$
9. $\langle I_1^R, O_1^R, B_2, B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L \rangle$
10. $\langle O_1^R, B_2, B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R \rangle$
11. $\langle B_2, B_0, O_0^L, I_0^L, I_0^R, O_0^R, B_1, O_1^L, I_1^L, I_1^R, O_1^R \rangle$
12. $\langle B_2, O_1^R, I_1^R, I_1^L, O_1^L, B_1, O_0^R, I_0^R, I_0^L, O_0^L, B_0 \rangle$
13. $\langle O_1^R, I_1^R, I_1^L, O_1^L, B_1, O_0^R, I_0^R, I_0^L, O_0^L, B_0, B_2 \rangle$
14. $\langle I_1^R, I_1^L, O_1^L, B_1, O_0^R, I_0^R, I_0^L, O_0^L, B_0, B_2, O_1^R \rangle$
15. $\langle I_1^L, O_1^L, B_1, O_0^R, I_0^R, I_0^L, O_0^L, B_0, B_2, O_1^R, I_1^R \rangle$
16. $\langle O_1^L, B_1, O_0^R, I_0^R, I_0^L, O_0^L, B_0, B_2, O_1^R, I_1^R, I_1^L \rangle$
17. $\langle B_1, O_0^R, I_0^R, I_0^L, O_0^L, B_0, B_2, O_1^R, I_1^R, I_1^L, O_1^L \rangle$
18. $\langle O_0^R, I_0^R, I_0^L, O_0^L, B_0, B_2, O_1^R, I_1^R, I_1^L, O_1^L, B_1 \rangle$
19. $\langle I_0^R, I_0^L, O_0^L, B_0, B_2, O_1^R, I_1^R, I_1^L, O_1^L, B_1, O_0^R \rangle$
20. $\langle I_0^L, O_0^L, B_0, B_2, O_1^R, I_1^R, I_1^L, O_1^L, B_1, O_0^R, I_0^R \rangle$
21. $\langle O_0^L, B_0, B_2, O_1^R, I_1^R, I_1^L, O_1^L, B_1, O_0^R, I_0^R, I_0^L \rangle$
22. $\langle B_0, B_2, O_1^R, I_1^R, I_1^L, O_1^L, B_1, O_0^R, I_0^R, I_0^L, O_0^L \rangle$

Figure 4.2: The list of all vectors of arena variables.

the arena variables that represents the vector $\vec{v}$. Observe that the predicate $\mathit{Prefix}_{\vec{v}}$ is falsified by a valuation q if and only if q arranges arena variables in the same way as vector $\vec{v}$ does.

In general, our intention is to model (see Definition 86) a boolean relation. Arena variables set in some order presented in Fig. 4.1 constitute some kind of arena. When some other variable is set strictly between O_0^L and O_0^R then we see its value as a boolean zero, and if some variable is set strictly between O_1^L and O_1^R then we see its value as a boolean one. We need also I_0^L, I_0^R, and I_1^L, I_1^R. Sometimes we want to say: 'If a variable x is equal to zero, then a variable y is also equal to zero'. Unfortunately, concerning positive temporal relations, due to the lack of negation and strict order, we are unable to write something like $(O_0^L < x < O_0^R) \rightarrow (O_0^L < y < O_0^R)$. Instead (observe that the negative occurence of the strict order is equivalent to a positive occurence of the weak order, as explained below) we write $(O_0^L < x < O_0^R) \rightarrow (I_0^L \leq y \leq I_0^R)$ and assure that I_0^L, I_0^R are always strictly between O_0^L, O_0^R. Similarly, we assure that I_1^L, I_1^R are always strictly between O_1^L, O_1^R. This is the general idea, but sometimes because of technical reasons we also use O_0^{L1}, O_0^{L2} etc.

At this point we probably owe the reader one more explanation. Sometimes, when we think it is intuitive, we use negation or implication in the definition of relations. Nevertheless, they should be treated just as notational shortcuts and all relations we claim to be positive temporal are indeed definable by conjunction, disjunction, and $\leq$. In particular, the relation $(O_0^L < x < O_0^R) \rightarrow (I_0^L \leq y \leq I_0^R)$ is a shortcut for $(O_0^L \geq x \vee x \geq O_0^R) \vee (I_0^L \leq y \leq I_0^R)$.

Concerning positive temporal relations closed under reversed cycles it is hard to say that some variable must be set on the left (or on the right) of the another variable. Far more natural is to say that a value of some variable is inside an interval set by values of other variables or outside such an interval. Moreover, the property of being a value inside must be preserved by all representations of this interval on the circle on Fig. 4.1. The first situation is implemented by predicates $\mathit{In}_{\vec{v}}$, and the second one by $\mathit{Out}_{\vec{v}}$ where $\vec{v} \in \mathit{Arenas}$. More precisely, we define $\mathit{In}_{\vec{v}}(x, y_1, y_2)$ equal to $((y_1 < y_2) \rightarrow (y_1 \leq x \leq y_2)) \wedge ((y_2 < y_1) \rightarrow (x \leq y_2 \vee x \geq y_1))$ if $\vec{v}$ is a cycle of the vector from item 1 and equal to $((y_1 < y_2) \rightarrow (x \leq y_1 \vee x \geq y_2)) \wedge ((y_2 < y_1) \rightarrow (y_2 \leq x \leq y_1))$ if $\vec{v}$ is a reversed-cycle of the vector from item 1. Similarly, we define $\mathit{Out}_{\vec{v}}(x, y_1, y_2)$ equal to $((y_1 < y_2) \rightarrow (x \leq y_1 \vee x \geq y_2)) \wedge ((y_2 < y_1) \rightarrow (y_2 \leq x \leq y_1))$ if $\vec{v}$ is a cycle of the vector $\vec{v_A}$; and equal to $((y_1 < y_2) \rightarrow (y_1 \leq x \leq y_2)) \wedge ((y_2 < y_1) \rightarrow (x \leq y_2 \vee x \geq y_1))$ if $\vec{v}$ is a reversed cycle of the vector $\vec{v_A}$.

Example 84. *In this example we wil try to convince the reader that the formula*

$$(\neg \mathit{Prefix}_{\vec{v}}) \rightarrow \mathit{In}_{\vec{v}}(I_0^R, O_0^R, O_0^L)$$

is true for all $\vec{v} \in \mathit{Arenas}$. First recall that if $\mathit{Prefix}_{\vec{v}}$ is falsified by some valuation q, then q orders arena variables in the same way as the vector $\vec{v}$. Note also that variables $\{I_0^R, O_0^R, O_0^L\}$ used in $\mathit{In}_{\vec{v}}$ are arena variables and therefore they occur in $\mathit{Prefix}_{\vec{v}}$.

As it was defined above, if $\vec{v}$ is a cycle of the vector $\vec{v_A}$, then $\mathit{In}_{\vec{v}}(I_0^R, O_0^R, O_0^L)$ is equivalent to $((O_0^L < O_0^R) \rightarrow (O_0^L \leq I_0^R \leq O_0^R)) \wedge ((O_0^R < O_0^L) \rightarrow (I_0^R \leq O_0^R \vee I_0^R \geq O_0^L))$. Observe that in the case of vectors 1–2 and 6–11 from Fig. 4.2 the first implication is true because both its antecedent and its consequent are true while the second implication is true because is antecedent is false. In the case of vectors 3–5 we have a symmetric situation: the first implication is true because its antecedent is false while the second implication is true because its consequent is true.

In turn, if $\vec{v}$ is a reversed cycle of the vector $\vec{v_A}$, then $In_{\vec{v}}(I_0^R, O_0^R, O_0^L)$ is equivalent to $((O_0^L < O_0^R) \rightarrow (I_0^R \leq O_0^L \vee I_0^R \geq O_0^R)) \wedge ((O_0^R < O_0^L) \rightarrow (O_0^R \leq I_0^R \leq O_0^L))$. Note that the consequent of the first implication is true and the antecedent of the second his false for vectors 19–21 while consequent of the second is true and antecendant of the first is false for vectors 12–18 and 22.

The positive temporal language Γ_{Circle} consists of three relations: *UImp*, *BImp*, and *Final*. Each relation $R \in \Gamma_{Circle}$ is of the form

$$\bigwedge_{v \in Arenas} \phi_{\vec{v}}^R \tag{4.1}$$

. By using this conjunction we assure that R is closed under all cycles and reversed cycles. We now define $\phi_{\vec{v}}^R$ for each $R \in \Gamma_{Circle}$ and each $\vec{v} \in Arenas$.

1. First of our relations is $\mathit{UImp}(\vec{v_A}, p, O^L, O^R, f, I^L, I^R)$ with

$$\phi_{\vec{v}}^{UImp} := \mathit{Prefix}_{\vec{v}} \vee \mathit{Out}_{\vec{v}}(p, O^L, O^R) \vee \mathit{In}_{\vec{v}}(f, I^L, I^R). \tag{4.2}$$

   The name *UImp* stands for unary implication. It is justified by the context in which we use it. If both $\mathit{Prefix}_{\vec{v}}$ and $\mathit{Out}_{\vec{v}}(p, O^L, O^R)$ are falsified, then $\mathit{In}_{\vec{v}}(f, I^L, I^R)$ must be satisfied. We use this relation to express the implication: 'if v represents different values in an appropriate order and p is a value in the interval from I^L to I^R, then f is also a value in this interval'.

2. We have also binary implication $\mathit{BImp}(\vec{v_A}, p_1, p_2, O^L, O^R, f, I^L, I^R)$ defined by the conjunction of:

$$\phi_{\vec{v}}^{BImp} := \mathit{Prefix}_{\vec{v}} \vee \mathit{Out}_{\vec{v}}(p_1, O^L, O^R) \vee \mathit{Out}_{\vec{v}}(p_2, O^L, O^R) \vee \mathit{In}_{\vec{v}}(f, I^L, I^R). \tag{4.3}$$

   If $\mathit{Prefix}_{\vec{v}}$ as well as $\mathit{Out}_{\vec{v}}(p_1, O^L, O^R)$ and $\mathit{Out}_{\vec{v}}(p_2, O^L, O^R)$ are falsified, then $\mathit{In}_{\vec{v}}(f, I^L, I^R)$ must be satisfied.

3. Finally there is $\mathit{Final}(\vec{v_A}, f_0, f_1)$ with

$$\phi_{\vec{v}}^{Final} := \mathit{Prefix}_{\vec{v}} \vee \mathit{Out}_{\vec{v}}(f_0, B_0, B_2) \vee \mathit{Out}_{\vec{v}}(f_1, B_0, B_2). \tag{4.4}$$

   We want to see it in the following way. If $\mathit{Prefix}_{\vec{v}}$ is falsified, then $\mathit{Out}_{\vec{v}}(f_0, B_0, B_2)$ or $\mathit{Out}_{\vec{v}}(f_1, B_0, B_2)$ must be satisfied.

Lemma 85. *The positive temporal language Γ_{Circle} is closed under weakly increasing, weakly decreasing, and half monotone surjections only.*

Proof. We show it for the relation *UImp*. Proofs for relations *BImp* and *Final* are similar.

We start by showing that *UImp* is closed under unary surjections only. By Theorem 20 and Corollary 25, it is enough to show a qc-definition of some equality positive, non-negative relation over *UImp*. Consider a relation $\mathit{UImp}'(\vec{v}_A)$ defined by $\mathit{UImp}(\vec{v}_A, I_0^L, O_0^L, O_0^R, O_1^R, I_1^L, I_1^R)$. Let us note, by (4.1), that $\mathit{UImp}'(\vec{v}_A)$ is equivalent to $\bigwedge_{\vec{v} \in Arenas} \mathit{Prefix}_{\vec{v}}$. Indeed, if $\mathit{Prefix}_{\vec{v}}$ is falsified for some $\vec{v} \in Arenas$, then both $\mathit{Out}_{\vec{v}}(I_0^L, O_0^L, O_0^R)$ and $\mathit{In}_{\vec{v}}(O_1^R, I_1^L, I_1^R)$ are falsified. Conversely, if $\mathit{Prefix}_{\vec{v}}$ is satisfied, then both *UImp* and *UImp'* are trivially satisfied. Further $\bigwedge_{\Pi \in S_{11}} \mathit{UImp}'(\Pi(\vec{v_A}))$

where S_{11} is the set of all permutations of 11 elements is equal to $\bigvee_{x \neq y, x, y \in Var_A} x = y$. This equality relation is, of course, positive and, by Lemma 24, non-negative. Hence we are done.

To prove that *UImp* is closed under all weakly increasing, weakly decreasing and weakly half-monotone surjections it is enough, by Lemma 73 and Lemma 70, to show that *UImp* is closed under all reversed cycles. Let Var_{UImp} denote the set of variables of *UImp*. Assume on the contrary that there exists a satisfying valuation $q_1 : Var_{UImp} \to \mathbb{Q}$ for *UImp*, falsifying valuation $q_2 : Var_{UImp} \to \mathbb{Q}$ and some reversed cycle $\Pi : \mathbb{Q} \to \mathbb{Q}$ such that $\Pi q_1 = q_2$. By the definition of *UImp*, we have that if some valuation does not satitisfy *UImp*, then this valuation does not satisfy $Prefix_{\vec{v}} \vee Out_{\vec{v}}(p, O^L, O^R) \vee In_{\vec{v}}(f, I^L, I^R)$ for some $\vec{v}$. Because $q_1 = \Pi^{-1} q_2$ and Π^{-1} is a cycle, there exists some arena vector $\vec{v'}$ among vectors in Fig. 4.2 such that q_1 falsifies $Prefix_{\vec{v'}}$. Using similar reasoning as in Example 84, we can show that q_1 falsifies also both $Out_{\vec{v'}}(p, O^L, O^R)$ and $In_{\vec{v'}}(f, I^L, I^R)$. In consequence q_1 falsifies some conjunct $Prefix_{\vec{v'}} \vee Out_{\vec{v'}}(p, O^L, O^R) \vee In_{\vec{v'}}(f, I^L, I^R)$ of the definition of *UImp*. Hence we have a contradiction with the assumption that q_1 satisfies *UImp*. □

4.2.2 The Proof of PSPACE-hardness of $QCSP(\Gamma_{Circle})$

The proof of PSPACE-hardness of $QCSP(\Gamma_{Circle})$ is based on the proof of PSPACE-hardness of $QCSP(\{x_1 \neq x_2 \vee x_1 = x_3\})$ from [5]. We define analogous notions and follow analogous reasoning.

Definition 86. *A relation $R \subseteq \{0,1\}^n$ is force definable if there exists a prenex formula*

$$\Phi_{R,f_0,f_1}(\vec{v_A}, O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}, x_1, \ldots, x_n) = \mathcal{Q}\phi$$

over Γ_{Circle} that satisfies all of the following.

1. *$\mathcal{Q}$ is a quantifier prefix and ϕ is a quantifier-free part.*

2. *The quantifier prefix $\mathcal{Q}$ contains f_0 and f_1 as its two last variables, and they are both existentially quantified.*

3. *The set of free variables is equal to $\{\vec{v_A}, O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}, x_1, \ldots, x_n\}$.*

4. *Let $\vec{t} \in \{0,1\}^n$ and let $\vec{v} \in Arenas$. Let variables from Var_A be set to satisfy $\neg(Prefix_{\vec{v}})$ and let variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ be set to satisfy*

    - *$In_{\vec{v}}(O_0^{L1}, B_0, O_0^L)$,*
    - *$In_{\vec{v}}(O_0^{R1}, O_0^R, B_1)$,*
    - *$In_{\vec{v}}(O_1^{L1}, B_1, O_1^L)$, and*
    - *$In_{\vec{v}}(O_1^{R1}, O_1^R, B_2)$.*

    Further, let x_k for $k \in \{1, \ldots, n\}$ are set to satisfy $In_v(x_k, I_i^L, I_i^R)$ iff $t_k = i$ for $i = 0, 1$. Then the sentence $\Phi' := \mathcal{Q}(\phi \wedge \neg(In_{\vec{v}}(f_0, I_0^L, I_0^R) \wedge In_{\vec{v}}(f_1, I_1^L, I_1^R)))$ is false iff $t \in R$.

5. *For all $\vec{v} \in Arenas$ if values of arena variables satisfy $Prefix_{\vec{v}}$, or free variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ are set to satisfy*

    - *$In_{\vec{v}}(O_0^{L1}, I_0^L, I_0^R)$,*
    - *$In_{\vec{v}}(O_0^{R1}, I_0^L, I_0^R)$,*

- $In_{\vec{v}}(O_1^{L1}, I_1^L, I_1^R)$, *or*
- $In_{\vec{v}}(O_1^{R1}, I_1^L, I_1^R)$*;*

then Φ' *is always true.*

6. *(motonicity) For any setting to the free variables of* Φ_{R,f_0,f_1}*, if the formula* Φ' *is true, then changing the value of any variable* x_i *to satisfy* $(Out_{\vec{v}}(x_i, O_0^{L1}, O_0^{R1}) \vee Out_{\vec{v}}(x_i, O_1^{L1}, O_1^{R1}))$ *preserves the truth of* Φ'*.*

As it was described in Section 2.2 we can see a sentence as a two-player game. The intuition behind Definition 86 is as follows. If free variables of Φ_{R,f_0,f_1} are set according to conditions from item 4 and $\vec{t} \in R$, then the universal player has a strategy to force the existential player to satisfy $In_{\vec{v}}(f_0, I_0^L, I_0^R)$ and $In_{\vec{v}}(f_1, I_1^L, I_1^R)$ where $\vec{v} \in Arenas$ and $Prefix_{\vec{v}}$ is falsified. But if $Prefix_{\vec{v}}$ is falsified for some $\vec{v} \in Arenas$ and the condition from item 5 is fulfilled, then the existential player is able to falsify $In_{\vec{v}}(f_0, I_0^L, I_0^R)$ or $In_{\vec{v}}(f_1, I_1^L, I_1^R)$.

Note that variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ are different from $O_0^L, O_0^R, O_1^L, O_1^R$. It may be somewhat confusing, but, as we will see, we need all these variables.

Lemma 87. *There exists a polynomial-time algorithm that, given a boolean circuit C as input, produces a force definition of the relation* R_C *containing, as tuples, exactly the satisfying assignments of the circuit.*

Proof. As shown in [5], it is possible to compute in polynomial time a primitive positive definition of R_C over the relations $\mathcal{N}$ and $\mathcal{P}$ defined as

$$\mathcal{N} = \{0,1\}^3 \setminus \{(1,1,1)\}, \mathcal{P} = \{0,1\}^3 \setminus \{(0,0,0)\}.$$

Now, let $\exists S_1 \dots \exists S_k C$ be a primitive positive definition of $R_C(X_1, \dots, X_n)$, where C is the conjunction of atomic formulas of the form $\mathcal{N}(U_1, U_2, U_3)$ and $\mathcal{P}(U_1, U_2, U_3)$ with $U_1, U_2, U_3 \in \{S_1, \dots, S_k, X_1, \dots, X_n\}$. Let $N_1, \dots, N_l$ denote atomic formulas of the form $\mathcal{N}(U_1, U_2, U_3)$ in C and $P_1, \dots, P_m$ atoms of the form $\mathcal{P}(U_1, U_2, U_3)$.

We now give a force-definition $\Phi_{R_C,f_0,f_1}(\vec{v_A}, O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}, x_1, \dots, x_n)$ of R_c. The quantifier prefix is as follows:

$$\forall s_1 \dots \forall s_k \exists n_1 \dots \exists n_l \exists t_1^n \dots \exists t_l^n \exists p_1 \dots \exists p_m \exists t_1^p \dots \exists t_m^p \ \exists f_0 \exists f_1$$

The quantifier-free part of our definition contains the following constraints.

1. For each constraint $N_i = \mathcal{N}(U_1, U_2, U_3)$ in C and each $j = 1, 2, 3$, we have a constraint *UImp*$(\vec{v_A}, x_d, O_0^{L1}, O_0^{R1}, n_i, I_0^L, I_0^R)$ if U_j is equal to X_d for some $1 \leq d \leq k$ or a constraint *UImp*$(\vec{v_A}, s_e, O_0^{L1}, O_0^{R1}, n_i, I_0^L, I_0^R)$ if U_j is equal to S_e for some $1 \leq e \leq n$.

2. For each constraint $P_i = \mathcal{P}(U_1, U_2, U_3)$ in C and for each $j = 1, 2, 3$, there is a constraint *UImp*$(\vec{v_A}, x_d, O_1^{L1}, O_1^{R1}, p_i, I_1^L, I_1^R)$ if U_j is equal to X_d for some $1 \leq d \leq k$ or a constraint *UImp*$(\vec{v_A}, s_e, O_1^{L1}, O_1^{R1}, p_i, I_1^L, I_1^R)$ if U_j is equal to S_e for some $1 \leq e \leq n$.

3. We have a constraint *UImp*$(\vec{v_A}, n_1, O_0^{L1}, O_0^{R1}, t_1^n, I_0^L, I_0^R)$ and one constraint of the form *BImp*$(\vec{v_A}, n_i, t_{i-1}^n, O_0^{L1}, O_0^{R1}, t_i^n, I_0^L, I_0^R)$ for each $i = 2, \dots, l$.

4. We have a constraint *UImp*$(\vec{v_A}, p_1, O_1^{L1}, O_1^{R1}, t_1^p, I_1^L, I_1^R)$ and one constraint of the form *BImp*$(\vec{v_A}, p_i, t_{i-1}^p, O_1^{L1}, O_1^{R1}, t_i^p, I_1^L, I_1^R)$ for each $i = 2, \dots, m$.

5. Finally, we have a constraint *UImp*$(\vec{v_A}, t^n_l, O^{L1}_0, O^{R1}_0, f_0, I^L_0, I^R_0)$ and a constraint *UImp*$(\vec{v_A}, t^p_n, O^{L1}_1, O^{R1}_1, f_1, I^L_1, I^R_1)$.

Now we claim that this is indeed a force-definition of R_C.

Assume that the arena variables are set to falsify $\mathit{Prefix}_{\vec{v}}$ for some $v \in \mathit{Arenas}$ and that values of $O^{L1}_0, O^{R1}_0, O^{L1}_1, O^{R1}_1$ satisfy conditions given in item 4 of Definition 86. Assume also that $\vec{t} \in \{0,1\}^n$ and x_k satisfies $\mathit{In}_{\vec{v}}(x_k, I^L_i, I^R_i)$ iff $\vec{t}_k = 1$ where $k = 1, \ldots, n$ and $i = 0, 1$.

We now claim that if $\vec{t} \in R_C$, then the universal player can enforce $\mathit{In}_{\vec{v}}(f_0, I^L_0, I^R_0)$ and $\mathit{In}_{\vec{v}}(f_1, I^L_1, I^R_1)$. To achieve that he first sets each variable s_j for $j = 1, \ldots, k$ to satisfy $\mathit{In}_{\vec{v}}(s_j, I^L_i, I^R_i)$ iff $S_j = i$ for $i = 0, 1$. Since $\vec{t} \in R_C$, all atoms N_i for $i = 1, \ldots, l$ and P_i for $i = 1, \ldots, m$ are satisfied. It means that among the three arguments of N_i at least one is false and at least one argument of P_i is true. Consider the case of N_i. The variable corresponding to the false argument is set to a value between I^L_0 and I^R_0, which is strictly between O^{L1}_0 and O^{R1}_0, so by constraints in item 1 the existential player is forced to set the value of n_i between I^L_0 and I^R_0, that is, to satisfy $\mathit{In}_{\vec{v}}(n_i, I^L_0, I^R_0)$. Similarly, by constraints in item 2 all variables p_i must be set between I^L_1 and I^R_1. Now, using simple induction and constraints in item 3 (and 4, respectively) one can show that if all variables $n_1, \ldots, n_i$ (resp.,$p_1, \ldots, p_i$) are strictly between O^{L1}_0 and O^{R1}_0 (resp.,O^{L1}_1 and O^{R1}_1), then the existential player is forced to set the value of t^n_i between I^L_0 and I^R_0 (resp., all t^p_i between I^L_1 and I^R_1). Finally by constraints in item 5 the existential player is forced to set f_0 to a value between I^L_0 and I^R_0 and f_1 to a value between I^L_1 and I^R_1.

Conversely, if $\vec{t} \notin R_C$, then the existential player is able to set at least one n_i where $1 \le i \le l$ to falsify $\mathit{In}_{\vec{v}}(n_i, I^L_0, I^R_0)$; or at least one p_i where $1 \le i \le m$ to falsify $\mathit{In}_{\vec{v}}(p_i, I^L_1, I^R_1)$. Note that he can do it independently of values the universal player sets to $s_1, \ldots, s_k$.

Therefore we are done with item 4 of Definition 86 and we can now turn to item 5. Note that if arena variables satisfy $\mathit{Prefix}_{\vec{v}}$ for each $\vec{v} \in \mathit{Arenas}$, then each clause in Φ_{R_C, f_0, f_1} is simply satisfied before the game starts. Therefore constraints listed above cannot force the existential player to satisfy either $\mathit{In}_{\vec{v}}(f_0, I^L_0, I^R_0)$ or $\mathit{In}_{\vec{v}}(f_1, I^L_1, I^R_1)$. Now, we consider a condition concerning variables $O^{L1}_0, O^{R1}_0, O^{L1}_1, O^{R1}_1$. Without loss of generality assume that $\mathit{In}_{\vec{v}}(O^{L1}_0, I^{L1}_0, I^{L2}_0)$ or $\mathit{In}_{\vec{v}}(O^{R1}_0, I^{L1}_0, I^{L2}_0)$ is satisfied. Then the existential player can set n_1 equal to O^{L1}_0 or to O^{R1}_0. Now the constraint *UImp* in item 3 is satisfied without forcing the existential player to set t^n_1 to satisfy $\mathit{In}_{\vec{v}}(t^n_1, I^L_0, I^R_0)$, so the existential player is not forced to set f_0 to satisfy $\mathit{In}_{\vec{v}}(f_0, I^L_0, I^R_0)$.

It remains to consider monotonicity. Note that due to constraints of Φ_{R_C, f_0, f_1} a variable x_i may be used by the universal player to enforce some move of the existential player only if $(\neg \mathit{Out}_{\vec{v}}(x_i, O^{L1}_0, O^{R1}_0) \vee \neg \mathit{Out}_{\vec{v}}(x_i, O^{L1}_1, O^{R1}_1))$ is true.

We now claim that our force definition of $R_C(X_1, \ldots, X_n)$ may be computed in polynomial time. By the remark at the beginning of the proof it is enough to show that it can be computed in polynomial time with respect to the size of its primitive positive definition $P_D := \exists S_1 \ldots \exists S_k C$. To see this observe that the quantifier prefix of Φ_{R_C, f_0, f_1} as well as constraints from each of items 1–5 may be constructed easily in polynomial time with respect to the size of P_D. □

Similarly as in [5] in case of $\{x_1 \neq x_2 \vee x_1 = x_3\}$ to prove Theorem 83, we reduce from *succinct graph unreachability*. In this problem, the input is a boolean circuit with $2c$ inputs that represent a graph G whose vertices are the tuples in $\{0,1\}^c$. There is a directed edge $(\vec{X}, \vec{Y})$ in the graph iff the circuit returns true given the input $(\vec{X}, \vec{Y})$. The question is to decide whether or not there is a directed path from $\vec{S}$ to $\vec{T}$. This problem is known to be *PSPACE*-complete.

Define $R_i \subseteq \{0,1\}^{2c}$ to be the relation containing exactly the tuples $(\vec{X}, \vec{Y})$ such that there exists a directed path in G from $\vec{X}$ to $\vec{Y}$ of length less than or equal to 2^i. Then there is a path in G from $\vec{S}$ to $\vec{T}$ iff $(\vec{S}, \vec{T}) \in R_c$.

From Lemma 87 it is not hard to infer that a force definition of R_0 is computable in polynomial time. Now, by induction we show that a force definition of R_c is also computable by a polynomial algorithm.

To prove Theorem 83 we will need the force definition of R_c but a *special force definition* of R_c. The notion of a special force definability of a binary relation is very closely related to the notion of a force definability. To introduce the notion of a special force definability of boolean relations it is enough to replace each occurrence of O_i^{L1} in Definition 86 with O_i^L and each each occurrence of O_i^{R1} with O_i^R for all $i = 1, 2$. In a similar way by replacing O_i^{L1} and O_i^{R1} with O_i^L and O_i^R for all $i = 1, 2$ we can transform a force definition of a boolean relation into its special force definition. However, to obtain a special force definition of R_c we will need a force definition of other relations.

Lemma 88. *A special force definition of the relation R_c is computable in polynomial time.*

Proof. (Part One) The proof goes by induction. For all $i \in \{0, \ldots, c\}$ we will give a special force definition Φ_{R_i, f_0, f_1} of the relation R_i.

To obtain Φ_{R_i, f_0, f_1} we need a special force definition of R_{i-1} and a force definition of the relation E equal to $\{\langle X_1, \ldots, X_{2c}, Y_1, \ldots, Y_{2c}\rangle \in \{0,1\}^{4c} \mid \forall i \in \{1, \ldots, 2c\}\ X_i = Y_i\}$.

Let now

$$\Phi_{E, f_0', f_1'}(\vec{v_A}, O_0^{L2}, O_0^{R2}, O_1^{L2}, O_1^{R2}, \vec{u}^1, \vec{u}^2, \vec{x}, \vec{z}) = Q'\phi' \tag{4.5}$$

and

$$\Phi_{E, f_0'', f_1''}(\vec{v_A}, O_0^{L2}, O_0^{R2}, O_1^{L2}, O_1^{R2}, \vec{u}^1, \vec{u}^2, \vec{z}, \vec{y}) = Q''\phi'' \tag{4.6}$$

be force representations for E where $O_0^{L2}, O_0^{R2}, O_1^{L2}, O_1^{R2}$ are fresh variables.

Let

$$\Phi_{R_{i-1}, f_0, f_1}(\vec{v_A}, O_0^L, O_0^R, O_1^L, O_1^R, \vec{x}, \vec{y}) = Q_i\phi_i \tag{4.7}$$

be a special force definition of R_{i-1}. As it was explained before the formulation of the lemma, in a special force definition variables $O_0^{L1}, O_0^{R1}, O_1^{L1}, O_1^{R1}$ are replaced with $O_0^L, O_0^R, O_1^L, O_1^R$

Before we show how to construct a special force definition for R_i, note that both definitions of E as well as a special force definition of R_0 can be constructed in polynomial time, in the way shown in the proof of Lemma 87.

Assume now that we have (4.5), (4.6), and (4.7). We show that

$$\Phi_{R_i, g_0, g_1}(\vec{v_A}, O_0^L, O_0^R, O_1^L, O_1^R, \vec{x}, \vec{y})$$

with a quantifier prefix

$$\forall \vec{z} Q' Q'' \forall \vec{u}^1 \forall \vec{u}^2 Q_i \exists O_0^{L2} \exists O_0^{R2} \exists O_1^{L2} \exists O_1^{R2} \exists g_0 \exists g_1$$

and a quantifier-free part that is a conjunction of

$$\phi' \wedge \phi'' \wedge \phi_i \tag{4.8}$$

and

$$\textit{UImp}(\vec{v_A}, f_0, O_0^L, O_0^R, O_0^{L2}, B_0, O_0^L) \wedge \textit{UImp}(\vec{v_A}, f_0, O_0^L, O_0^R, O_0^{R2}, O_0^R, B_1) \wedge$$
$$\textit{UImp}(\vec{v_A}, f_1, O_1^L, O_1^R, O_1^{L2}, B_1, O_1^L) \wedge \textit{UImp}(\vec{v_A}, f_1, O_1^L, O_1^R, O_1^{R2}, O_1^R, B_2) \quad (4.9)$$

and

$$\textit{BImp}(\vec{v_A}, f_0', f_0'', O_0^L, O_0^R, g_0, I_0^L, I_0^R) \wedge \textit{BImp}(\vec{v_A}, f_1', f_1'', O_1^L, O_1^R, g_1, I_1^L, I_0^R) \quad (4.10)$$

where $\vec{z}, \vec{u}^1, \vec{u}^2, g_0, g_1$ are fresh variables, is a force definition of R_i.

In the following ((Part Two)–(Part Five)), we prove that this is indeed a special force definition of R_i. To prove it we will follow the items 4–6 of Definition 86. Items 1–3 are straightforward to check.

(Part Two) First assume that arena variables are set to falsify $\textit{Prefix}_{\vec{v}}$ for some $\vec{v} \in \textit{Arenas}$. Because O_i^{L1} is equal to O_i^L and O_i^{R1} is equal to O_i^R for $i = 0, 1$, conditions from item 4 of Definition 86 concerning these variables are satisfied. Assume also that $\langle \vec{X}, \vec{Y} \rangle$ is in R_i, that is, there is a path of length at most 2^i from $\vec{X}$ to $\vec{Y}$. Let variables x_k, y_l for $1 \le k, l \le c$ be set to satisfy $\textit{In}_{\vec{v}}(x_k, I_j^L, I_j^R)$ iff $X_k = j$ and $\textit{In}_{\vec{v}}(y_l, I_j^L, I_j^R)$ iff $Y_k = j$ where $j = 0, 1$. In this case, as we show, the universal player has a strategy to satisfy $\textit{In}_{\vec{v}}(g_0, I_0^L, I_0^R)$ and $\textit{In}_{\vec{v}}(g_1, I_1^L, I_1^R)$. If $R_i(\vec{X}, \vec{Y})$ is true, then there exists $\vec{Z} \in \{0,1\}^c$ such that $R_{i-1}(\vec{X}, \vec{Z})$ and $R_{i-1}(\vec{Z}, \vec{Y})$ are true. At the beginning of the game, the universal player sets z_k for $1 \le k \le c$ to satisfy $\textit{In}_{\vec{v}}(z_k, I_j^L, I_j^R)$ iff $Z_i = j$. Then the players play on Q' and Q''. We now show that if the existential player does not satisfy at least one of $\textit{In}_{\vec{v}}(f_0', I_0^L, I_0^R)$, $\textit{In}_{\vec{v}}(f_0'', I_0^L, I_0^R)$, $\textit{In}_{\vec{v}}(f_1', I_1^L, I_1^R)$, $\textit{In}_{\vec{v}}(f_1'', I_1^L, I_1^R)$; then the universal player can falsify Φ_{R_i, g_0, g_1}. Assume the contrary. Without loss of generality suppose that after evaluating variables from Q' and Q'' the formula defining $\textit{In}_{\vec{v}}(f_0', I_0^L, I_0^R)$ is not satisfied. Then the universal player can set $\vec{u}^1$ equal to the values of $\vec{x}$ and $\vec{u}^2$ equal to the values of $\vec{z}$. Because $R_{i-1}(\vec{x}, \vec{z})$ is true, we have that $Q_i\ \phi_i$ forces the existential player to satisfy both $\textit{In}_{\vec{v}}(f_0, I_0^L, I_0^R)$. and $\textit{In}_{\vec{v}}(f_1, I_1^L, I_1^R)$. Then, by (4.9) she must satisfy all of the following: $\textit{In}_{\vec{v}}(O_0^{L2}, B_0, O_0^L)$, $\textit{In}_{\vec{v}}(O_0^{R2}, O_0^R, B_1)$, $\textit{In}_{\vec{v}}(O_1^{L2}, B_1, O_0^L)$, and $\textit{In}_{\vec{v}}(O_1^{R2}, O_1^R, B_2)$. Consider now the formula $\Phi_{E, f_0', f_1'}$. Because variables $\vec{u}^1$ and $\vec{x}$ as well as $\vec{u}^2$ and $\vec{z}$ are set to the same values, we have that the relation E holds for $\langle u_1^1, \ldots, u_c^1, x_1, \ldots, x_c, u_1^1, \ldots, u_c^1, z_1, \ldots, z_c \rangle$. Therefore by item 4 of Definition 86 applied to $\Phi_{E, f_0', f_1'}$ the existential player has to satisfy both $\textit{In}_{\vec{v}}(f_0', I_0^L, I_0^R)$ and $\textit{In}_{\vec{v}}(f_1', I_1^L, I_1^R)$. Hence we have a contradiction with the assumption. Further, if $\textit{In}_{\vec{v}}(f_0', I_0^L, I_0^R)$, $\textit{In}_{\vec{v}}(f_0'', I_0^L, I_0^R)$, $\textit{In}_{\vec{v}}(f_1', I_1^L, I_1^R)$, and $\textit{In}_{\vec{v}}(f_1'', I_1^L, I_1^R)$ are satisfied, then, by (4.10) we get that $\textit{In}_{\vec{v}}(g_0, I_0^L, I_0^R)$ and $\textit{In}_{\vec{v}}(g_1, I_1^L, I_1^R)$ are satisfied as well.

Now assume that $\langle \vec{X}, \vec{Y} \rangle \notin R$. (We are still in item 4 of Definition 86.) In this case we will show that the existential player is not obliged to satisfy both $\textit{In}_{\vec{v}}(g_0, O_0^L, O_0^R)$ and $\textit{In}_{\vec{v}}(g_1, O_1^L, O_1^R)$. Since $\langle \vec{X}, \vec{Y} \rangle \notin R$, there exists no $\vec{Z}$ that satisfies $R_{i-1}(\vec{X}, \vec{Z})$ and $R_{i-1}(\vec{Z}, \vec{Y})$ at the same time. Take any vector $\vec{Z}$; without loss of generality we can assume that $\langle \vec{X}, \vec{Z} \rangle \notin R_{i-1}$. If the universal player sets $\vec{u}^1$ to the values of $\vec{x}$ and $\vec{u}^2$ to the values of $\vec{z}$, then the existential player can win ϕ_i, that is, she is not obliged to satisfy both $\textit{In}_{\vec{v}}(f_0, I_0^L, I_0^R)$ and $\textit{In}_{\vec{v}}(f_1, I_1^L, I_1^R)$. By (4.9), she can satisfy at least one of the following: $\textit{In}_{\vec{v}}(O_0^{L2}, I_0^L, I_0^R)$, $\textit{In}_{\vec{v}}(O_0^{R2}, I_0^L, I_0^R)$, $\textit{In}_{\vec{v}}(O_1^{L2}, I_1^L, I_1^R)$, or $\textit{In}_{\vec{v}}(O_1^{R2}, I_1^L, I_1^R)$. Therefore, by item 5 of Definition 86, playing on Q', she can falsify $\textit{In}_{\vec{v}}(f_0', I_0^L, I_0^R)$ or $\textit{In}_{\vec{v}}(f_1', I_1^L, I_1^R)$. Hence, by (4.10) she is not obliged to satisfy both $\textit{In}_{\vec{v}}(g_0, O_0^L, O_0^R)$ and $\textit{In}_{\vec{v}}(g_1, O_1^L, O_1^R)$. In turn, if the universal player sets $\vec{u}^1$ to the values of $\vec{z}$ and $\vec{u}^2$ to the values of $\vec{y}$, then he has no power in ϕ' and the existential player can easily falsify $In_{\vec{v}}(f'_0, I^L_0, I^R_0)$ or $In_{\vec{v}}(f'_1, I^L_1, I^R_1)$. Once again, by (4.10), this means that she is not obliged to satisfy both $In_{\vec{v}}(g_0, O^L_0, O^R_0)$ and $In_{\vec{v}}(g_1, O^L_1, O^R_1)$. All other cases are similar: whatever the universal player sets to $\vec{u}^1$ and $\vec{u}^1$, the existential player wins either on ϕ', on ϕ'', or on ϕ_i.

(Part Three) In this part of the proof we examine whether Φ_{R_i,g_0,g_1} is stated in accordance with item 5 of Definition 86. First note each clause of Φ_{R_i,g_0,g_1} contains $\mathit{Prefix}_{\vec{v}}$ for some $\vec{v} \in \mathit{Arenas}$. Hence if some valuation of arena variables satisfies Prefix_v for all vectors $\vec{v}$, then each conjunct of Φ_{R_i,g_0,g_1} is already satisfied. The second thing to observe is that we provide a special force definition of R_i. Therefore Φ_{R_i,g_0,g_1} unifies variables $O^{L1}_0, O^{R1}_0, O^{L1}_1, O^{R1}_1$ from Definition 86 with variables $O^L_0, O^R_0, O^L_1, O^R_1$. Hence if at least one of the four formulas listed in item 5 of Definition 86 is satisfied, then the valuation of arena variables satisfies $\mathit{Prefix}_{\vec{v}}$ for all vectors of arena variables $\vec{v}$.

(Part Four) The definition of R_i is also monotone. Observe that the universal player has no business in setting any universal variable v_u to satisfy $Out_{\vec{v}}(v_u, O^L_0, O^R_0)$ or $Out_{\vec{v}}(v_u, O^L_1, O^R_1)$ for any vector of arena variables $\vec{v}$. He cannot enforce any move of the existential player in this way. To see this, it is enough to examine the definitions of *UImp* and *BImp* once again.

(Part Five) We now show that the force definition of R_c is computable in polynomial time with respect to the size of the circuit representing a graph G. By Lemma 87 we have that $\Phi_{E,f'_0,f'_1}, \Phi_{E,f''_0,f''_1}$ and Φ_{R_0,f_0,f_1} can be computed in polynomial time, in particular, the quantifier prefixes Q' and Q'' are of polynomial size and do not depend on i. Constructing a quantifier-free part of Φ_{R_c,f_0,f_1} we need $\Phi_{E,f'_0,f'_1}, \Phi_{E,f''_0,f''_1}$ as well as six atoms given in (4.9) and (4.10) on each level $i \in \{1, \ldots, c\}$. Concerning the quantifier prefix Q_i we need Q_{i-1}, Q', Q'' and $3 * c + 6$ variables including: $\vec{z}, \vec{u^1}, \vec{u^2}, g_0, g_1, O^{L2}_0, O^{R2}_0, O^{L2}_1, O^{R2}_1$. Therefore, the whole construction of Φ_{R_c,f_0,f_1} may be done in polynomial time with respect to the size of the circuit representing G. □

Proof. (of Theorem 83) Let $\Phi_{R_c,g_0,g_1}(\vec{v_A}, O^L_0, O^R_0, O^L_1, O^R_1, \vec{x}, \vec{y}) = Q_c\phi_c$ be a force definition of R_c. We use it now to give an instance of *QCSP*(Γ_{Circle}) that is true if and only if there is no path from $\vec{S}$ to $\vec{T}$ in the succinctly represented graph.

The instance created is

$$\forall \vec{v_A} Q_c\phi_c \wedge \vec{x} = \vec{s} \wedge \vec{y} = \vec{t} \wedge \mathit{Final}(\vec{v_A}, g_0, g_1)$$

where $\vec{s}_i = \vec{t}_j = I^L_k$ if $\vec{S}_i = \vec{T}_j = k$ for all $1 \leq i, j \leq c$ and $k = 0, 1$.

The universal player starts the game. To have a chance to win (to falsify) the sentence he must set arena variables to falsify $\mathit{Prefix}_{\vec{v}}$ for some $\vec{v} \in \mathit{Arenas}$. Otherwise each clause from ϕ_c is satisfied already at the beginning. If there is a path from s to t, then the universal player can enforce the existential player to satisfy $In_{\vec{v}}(g_0, I^L_0, I^R_0)$ and $In_{\vec{v}}(g_1, I^L_1, I^R_1)$. It is not hard to verify that it contradicts $\mathit{Final}(\vec{v_A}, g_0, g_1)$ — see (4.4). If $(s,t) \notin R^c$, then the existential player can satisfy $Out_{\vec{v}}(g_0, B_0, B_2)$ or $Out_{\vec{v}}(g_1, B_0, B_2)$; and in consequence satisfy $\mathit{Final}(\vec{v_A}, g_0, g_1)$. □

4.3 Proof of the Main Theorem and Meta-Problems

Eventually, we prove Theorem 6, which is the main contribution of this thesis, and hence we solve the Research Problem 1 for quantified constraints satisfaction problems over positive temporal relations. We also consider here two 'metaproblems' — see Research Problem 2. Each of them is considered with respect to different representations of positive temporal relations. First of these representations is provided by logical formulas implied by filters whereas the second one uses orbits of tuples.

Lemma 89. *Let* Γ *be a positive temporal language such that each its relation is definable as a conjunction of weak inequalities but not as a conjunction of equalities. Then* *QCSP*(Γ) *is NLOGSPACE-complete.*

Proof. In [7] it is shown that each positive temporal language Γ from case 2 of Theorem 6. is decidable in NLOGSPACE. To prove Theorem 6 we need hardness as well. One of the most famous NLOGSPACE-complete problems is the problem of reachability in a directed graph [37]. We are given a digraph G and two vertices s and t, and we ask whether there exists a directed path from s to t. We will need some version of this problem, which we call THISWAYORTHATWAYȦn instance of this problem, once again, consists of a digraph G and two vertices s, t. However, this time we ask whether there exists a directed path from s to t, or from t to s. First, we wil show that THISWAYORTHATWAY is NLOGSPACE-hard by reducing reachability to this problem, and then we use THISWAYORTHATWAY to prove the lemma.

(Part One) Let G, s, t be an instance of the reachability problem. The reduction to THISWAYORTHATWAY goes as follows. Let G' be the digraph G with one extra vertex v and one extra arc (t, v). We will now show that t is reachable from s in G if and only if there is a directed path from s to v or a directed path form v to s in G'. Indeed, if there exists a directed path from s to t, then we can prolong it with the arc (t, v) and obtain a directed path from s to v in G'. On the other hand, there is no directed path from v to s in G'. Hence assume that there exists a directed path from s to v. Since it must go through t, the vertex t is reachable from s. This reduction can be of course provided in logarithmic space. Concerning membership in NLOGSPACE, it is enough to reduce THISWAYORTHATWAY to reachability.

(Part Two) We now give a logarithmic space reduction from the complement of the problem THISWAYORTHATWAY to *QCSP*(Γ) where Γ is an arbitrary positive temporal language whose relations are definable as a conjunction of inequalities, but not definable as a conjunction of equalities. Recall that NLOGSPACE is closed under complementation [37]. Let $R \in \Gamma$ be a relation that can be defined as a conjunction of inequalities bot cannot be defined as a conjunction of equalities, let $\phi_R(x_1, \ldots, x_n)$ equal to $\wedge_{i=1}^{m} y_i \leq z_i$ be a definition of R. Since R is not definable as a conjunction of equalities, the formula ϕ_R contains some conjunct $y_i \leq z_i$ where $1 \leq i \leq n$ such that $z_i \leq y_i$ is not in ϕ_R. Let $\{w_1, \ldots, w_{n-2}\} = \{x_1, \ldots, x_n\} \setminus \{y_i, z_i\}$. Define the relation $R_{arc}(y_i, z_i, w_1, \ldots, w_{n-2})$ as $R(x_1, \ldots, x_n)$, that is, by permuting columns of R. Observe that the relation $\exists w_1 \ldots \exists w_{n-2}\ R(x_1, \ldots, x_n)$ is just the order $\leq$, but we cannot use it directly because we want all quantifiers in the final formula to appear in the quantifier prefix.

(Part Three) Let G be a digraph and s, t its vertices. We now give an instance ψ of *QCSP*(Γ) that is not valid if and only if there is a directed path from s to t or a directed path from t to s in G. Let ψ consists of a quantifier prefix Q and a quantifier-free part ϕ. The reduction is as follows. We introduce a variable v for each of vertices of G. Furthermore, for each arc (v_1, v_2), the quantifier-free formula ϕ contains a conjunct $R_{arc}(v_1, v_2, q_1, \ldots, q_{n-2})$ where $q_1, \ldots, q_{n-2}$ are fresh variables. For each such an arc the quantifier prefix Q contains a sequence $\exists q_1, \ldots, \exists q_{n-2}$. These sequences come in an arbitrary order but they all occur at the end of the quantifier prefix. The beginning of the quantifier prefix is $\exists s \forall t \exists p_1 \dots \exists p_k$ where $p_1, \dots, p_k$ are all vertices of G except for s and t.

(Part Four) It is easily seen that this reduction may be obtained in logarithmic memory. To see that the reduction is correct, note the following. If t is reachable from s, then it is enough for the universal player to set to the variable t something strictly less than that what is set to s. Independently of what we set to the remaining variables, the sentence ψ is already false. Likewise, if there is a path from t to s, then the universal player can set to t something strictly greater than the value that is set to s.

We will now show that if neither t is reachable from s nor s from t, then it is enough for the existential player to set the variables $p_1, \dots, p_k$ in the following way. To each variable that is in some cycle with s (respectively t) we set the value of s (respectively t). To all remaining variables from which either t or s is reachable, we set minimum of the values of t and s. Likewise, to all remaining variables that can be reached from either s or t we set maximum of values of s or t. Observe that some valuation q of variables satisfies ϕ if provided v_1 is reachable from v_2 we have that $q(v_1) \geq q(v_2)$ and that our valuation sets to all variables reachable from s or t a value which is greater than the value of both s and t. Similarly, if there is a path from some vertex v to t or s, then its value is less than both of the values of these variables. Because there is neither a path from s to t nor from t to s, there are not vertices between s and t.

Finally, we will show how to satisfy each conjunct of ϕ in the case where s and t are not connected to each other. Since the relation R_{arc} is not empty and the values of v_1 and v_2 do not falsify $R_{arc}(v_1, v_2, q_1, \dots, q_{n-2})$, we can always find the satisfying values for $q_1, \dots, q_{n-2}$. Note that each such $n-2$-tuple of variables occur only in one conjunct of ϕ. □

Proof. (of Theorem 6) Theorem 29 gives us a partition of positive temporal languages into classes containing templates preserved by the same surjective polymorphisms. Moreover, the same theorem gives us a logical representation for languages from families 1–5 of Theorem 6.

Complexity characterization of qcsps for languages whose relations are definable as the conjunction of equalities was considered in [5] — see also Section 1.4. As it was established there, these problems belong to LOGSPACE. Lemma 89 gives us the complexity characterization of item 2. The families of qcsps from items 3 and 4 are examined in Section 4.1. Theorem 81 states that former languages give rise to P-complete qcsps problems; the latter ones are dual and can be proved to be P-complete in a similar way. To give the complexity characterization for item 5 we refer the reader once again to [5] and Section 1.4.

What remains to show is that families of languages from items 6–9 of Theorem 29 give rise to PSPACE-complete quantified constraint satisfaction problems. Note that this is what is proved by Proposition 82 and Theorem 83 in Section 4.2. □

Meta-problems Theorem 6 provides the complete complexity classification for quantified constraint satisfaction problems over positive temporal templates. Consider a version of Research Problem 2 where 'csps' is replaced with 'qcsps' and Γ is the set of positive temporal templates. We consider two different versions of this problem each of which concerns different representations of positive temporal relations.

If each relation of a given constraint language Γ is given by a simple qc-formula in the style preferred by the formulation of Theorem 6, then we can easily say to which of the classes 1–6 the language Γ belongs. As noted in Section 1.5, it can be done in linear time.

The other representation of temporal relation we should discuss is that provided by orbits of tuples — see Example 12 from Section 2.2. As we see, the algebraic approach will appear useful here.

Theorem 90. *Let* Γ *be a positive temporal language each of which relation is represented by a set of orbits of tuples. Then the meta-problem is decidable in polynomial time.*

Proof. Theorem 29 gives a characterization of families of languages in terms of surjective polymorphisms, that is, each such family contains at least one surjective polymorphism that does not preserve others. Therefore to prove the theorem, it is enough to show that for a given temporal relation R we can check whether it is closed under some rational function f in polynomial time. Because there are only several functions of interest, we can assume that f is not a part of the instance. Therefore to check whether R is preserved by some function of arity k mentioned in the formulation of Theorem 29, it is enough to check for all of tuples $t_1, \ldots, t_k$ of R whether $f(t_1, \ldots, t_k)$ belongs to R. Note that it can be done time $O(|R|^k)$ where $|R|$ is the size of R. □

Chapter 5

Conclusions

In this dissertation, we have provided a complete complexity characterization of quantified constraint satisfaction problems for positive temporal languages. By means of the polymorphism based algebraic approach we have first classified these languages according to their surjective polymorphisms, and then using the surjective preservation theorem, with respect to their positive first order definability and the complexity of their qcsps. Besides the algebraic approach we have been using other methods including game-based approach to verification of truth of first order sentences as well as the filter representation of temporal relations.

Positive temporal relations have robust logical and algebraical characterizations. A temporal relation is positive if and only if it has a positive first order definition over the order of rational numbers, and from the algebraic side, if and only if it is closed under all weakly increasing unary surjections on $\mathbb{Q}$. Nevertheless the structure, in the sense of containing tuples, of an arbitrary temporal relation is significantly more varied than a structure of a positive temporal relation. Thus it is natural to ask whether a similar complexity characterization theorem exists for all temporal languages.

More generally, temporal languages are ω-categorical structures. Since the surjective preservation theorem is general enough it is tempting to look for characterization of qcsps for other classes of ω-categorical structures and for a characterization of qcsps for ω-categorical languages in general. However, it will be very difficult. Let us recall that in the case of finite domains the characterization of qcsps is not known even for a three element universe. Therefore in the case of the general classification partial results might be satisfactory. It also means that it is interesting to compare known tractable families of finite-domain and ω-categorical languages — see for example [6].

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