Resolution Decision Procedures for Modal Logics

Vienna, 3 April 2006

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Topic of this talk

Decision Procedures for Modal Logics based on Translation into the Guarded Fragment.

- We introduce the guarded fragment.
- Explain how modal logics K, B can be translated into the guarded fragment.
- Explain why many other modal logics cannot be translated into the guarded fragment.
- Propose an improved translation with which most modal logics can be translated into the guarded fragment.
- Explore the borders of the improved translation method.

Motivation

- In my view, modal logics as such are not terribly important for applications, but the principles that underly them, are important.
- Modal logics are essentially isomorphic to description logics, which are used for modelling the structure of ontologies.
- Modal logics with transitive frames are related to temporal logics, and spatial logics.

The Guarded Fragment of First-Order Logic

Definition: A formula F is in the guarded fragment if

1. F is function free,
2. every quantification in F has form $\forall \overline{x}\ a \rightarrow G$, or $\exists \overline{x}\ a \wedge G$, where a is an atom, s.t. all free variables of G occur in a.

Introduced by Johan van Benthem, Hainal Andréka, and István Németi in 1998, with the intention of determining 'the modal fragment of first-order logic'.

First major decidable fragment since 1975.

The guarded fragment can be decided in DEXPTIME. (In EXPTIME when signature is fixed)

Multimodal Propositional Logic

Let PROP be the set of propositional variables, let Σ be some alphabet. Modal formulas are recursively defined as follows:

- a propositional variable $p \in \mathrm{PROP}$ is a modal formula,
- if ψ is a modal formula, then $\neg\psi$ is also a modal formula,
- if ψ_1, ψ_2 are modal formulas, then $\psi_1 \vee \psi_2,\ \psi_1 \wedge \psi_2,\ \psi_1 \rightarrow \psi_2,\ \psi_1 \leftrightarrow \psi_2$ are also modal formulas,
- if ψ is a modal formula, and $a \in \Sigma$, then $[a]\ \psi$ and $\langle a \rangle\ \psi$ are also modal formulas.

Kripke Semantics for Multimodal Logic

Modal formulas are interpreted in Kripke interpretations. An Kripke interpretation $\mathcal{M} = (W, R, V)$ is an ordered triple, of which W is a non-empty set, R is a function that assigns to each $a \in \Sigma$ a subset of $W \times W$, and V is a function that assigns to each $p \in \mathrm{PROP}$ a subset of W.

If $w \in W$, then

- $\mathcal{M}, w \models p$ iff $w \in R(p)$,
- $\mathcal{M}, w \models \neg\, \psi$ iff not $\mathcal{M}, w \models \psi$,
- $\mathcal{M}, w \models \psi_1 \vee \psi_2$ iff $\mathcal{M}, w \models \psi_1$ or $\mathcal{M}, w \models \psi_2$,
- $\mathcal{M}, w \models [a]\ \psi$ iff for all $w' \in W$, if $(w, w') \in R(a)$, then $\mathcal{M}, w' \models \psi$,
- $\mathcal{M}, w \models \langle a \rangle\ \psi$ iff there is a $w' \in W$, s.t. $(w, w') \in R(a)$, and $\mathcal{M}, w' \models \psi$.

Characterizations of Modal Logics

Different modal logics are characterized by different subsets of Kripke interpretations:

Logic K	(all Kripke interpretations)
Logic B	For $a \in \Sigma$, relations $R(a)$ are symmetric,
Logic $S4$	For $a \in \Sigma$, relations $R(a)$ are transitive and reflexive,
Logic $K4$	For $a \in \Sigma$, relations $R(a)$ are transitive,
Logic $S5$	For $a \in \Sigma$, relations $R(a)$ are equivalence relations.

Modal Logic and First-Order Logic:

We assume that there is a unique unary predicate symbol $\mathbf{P}$ associated to every $p \in \mathrm{PROP}$, and that there is a unique relation symbol $\mathbf{R}_a$ associated to every $a \in \Sigma$.

- To every Σ-interpretation (W, R, V), one can associate a first-order interpretation (W, V') as follows: For each $a \in \Sigma$, $V'(\mathbf{R}_a) = R(a)$, and for each $p \in \mathrm{PROP}$, $\quad V'(\mathbf{P}) = V(p)$.

  We call (W, V') the first-order interpretation associated to (W, R, V).

- To every first-order interpretation (W, V'), interpreting the $\mathbf{P}$ and $\mathbf{R}_a$, one can associate a Σ-interpretation (W, V) as follows: For each $a \in \Sigma$, $\quad R(a) = V'(\mathbf{R}_a)$, and for each $p \in \mathbf{PROP}$, $V(p) = V'(\mathbf{P})$. We call (W, R, V) the Kripke interpretation associated to (W, V').

Relational Translation (with recycling of variables):

Using the correspondence between the modal language and the first-order language, one can define:

$t(p, \alpha, \beta) = \mathbf{P}(\alpha)$,
$t(\neg\psi, \alpha, \beta) = \neg\, t(\psi, \alpha, \beta)$,
$t(\psi_1 \wedge \psi_2) = t(\psi_1, \alpha, \beta) \wedge t(\psi_2, \alpha, \beta)$,
$t(\psi_1 \vee \psi_2) = t(\psi_1, \alpha, \beta) \vee t(\psi_2, \alpha, \beta)$,
$t(\psi_1 \rightarrow \psi_2) = t(\psi_1, \alpha, \beta) \rightarrow t(\psi_2, \alpha, \beta)$,
$t(\psi_1 \leftrightarrow \psi_2) = t(\psi_1, \alpha, \beta) \leftrightarrow t(\psi_2, \alpha, \beta)$,
$t(\langle a \rangle\ \psi, \alpha, \beta) = \exists\beta\ [\ \mathbf{R}_a(\alpha, \beta) \wedge t(\psi, \beta, \alpha)\ ]$,
$t([a]\ \psi, \alpha, \beta) = \forall\beta\ [\ \mathbf{R}_a(\alpha, \beta) \rightarrow t(\psi, \beta, \alpha)\ ]$.

Theorem: Let $\mathcal{M}$ be a Σ-interpretation, let $\mathcal{M}'$ be a first-order interpretation, s.t. $\mathcal{M}$ and $\mathcal{M}'$ are associated interpretations of each other. Then $\mathcal{M}, w \models \psi$ iff $\mathcal{M}'[w \leftarrow \alpha] \models t(\psi, \alpha, \beta)$.

Theorem: Let $\mathcal{L}$ be a modal logic, defined by some restriction on Kripke frames.

Let $F_{\mathcal{L}}$ be a closed first-order formula, s.t. for each pair of associated interpretations $\mathcal{M}, \mathcal{M}'$, we have $\mathcal{M}' \models F_{\mathcal{L}}$ iff $\mathcal{M}$ is a Kripke interpretation allowed by the logic $\mathcal{L}$.

Let ψ be a modal formula. Then ψ is satisfiable in $\mathcal{L}$ iff $F \wedge \exists\alpha\ t(\psi, \alpha, \beta)$ is satisfiable in FOL.

Deciding Modal Logic through the Guarded Fragment

Theorem: For each modal formula ψ, $t(\psi, \alpha, \beta)$ is in the guarded fragment.

In this way, we obtain a nice way of deciding satisfiability in modal logic K and some other modal logics:

If one wants to decide whether formula ψ is satisfiable in modal logic $\mathcal{L}$, then test satisfiability of $F_{\mathcal{L}} \wedge \exists\alpha\ t(\psi, \alpha, \beta)$.

The method works as long as $F_{\mathcal{L}}$ is in the guarded fragment.

Unfortunately, many (simple) modal logics $\mathcal{L}$ have $F_{\mathcal{L}}$ outside of the guarded fragment. An example is S4, which is characterized by transitivity:

$$\forall w_1 w_2 w_3 \; \mathbf{R}_a(w_1, w_2) \land \mathbf{R}_a(w_2, w_3) \rightarrow \mathbf{R}_a(w_1, w_3).$$

In the rest of this talk:

- I give an alternative translation that translates formulas of modal logic S4 into the guarded fragment.
- I will show that the existence of this translation depends on the fact that S4 has a regular frame property.
- I will try to characterize the borders of the method.

Example: Modal Logic S4

A formula is in negation normal form if

it does not contain $\leftrightarrow$ and $\rightarrow$, and $\neg$ is applied only on propositional variables.

Improved Relational Translation for S4:

$$
\begin{aligned}
&t_{S4}(p, \alpha, \beta) = \mathbf{P}(\alpha),\\
&t_{S4}(\neg p, \alpha, \beta) = \neg\mathbf{P}(\alpha),\\
&t_{S4}(\psi_1 \wedge \psi_2) = t_{S4}(\psi_1, \alpha, \beta) \wedge t_{S4}(\psi_2, \alpha, \beta),\\
&t_{S4}(\psi_1 \vee \psi_2) = t_{S4}(\psi_1, \alpha, \beta) \vee t_{S4}(\psi_2, \alpha, \beta),\\
&t_{S4}(\langle a\rangle\ \psi, \alpha, \beta) = \exists\beta\ \mathbf{R}_a(\alpha, \beta) \wedge t_{S4}(\psi, \beta, \alpha),\\
&t_{S4}([a]\ \psi, \alpha, \beta) =\\
&\qquad X(\alpha)\ \wedge\\
&\qquad \forall\alpha\beta\ \ X(\alpha) \rightarrow \mathbf{R}_a(\alpha, \beta) \rightarrow X(\beta)\wedge \qquad\qquad (X \text{ is a new symbol})\\
&\qquad \forall\alpha\ \ X(\alpha) \rightarrow t_{S4}(\psi, \alpha, \beta).
\end{aligned}
$$

For every modal formula ψ in NNF, $t_{S4}(\psi, \alpha, \beta)$ is in the guarded fragment.

Theorem: For every modal formula ψ in NNF, ψ is S4-satisfiable iff $t_{S4}(\psi, \alpha, \beta)$ is FO-satisfiable

proof: $(\Rightarrow)$
if ψ is S4-satisfiable, then ψ has a Σ-model $\mathcal{M} = (W, R, V)$, in which each R_a is transitive and reflexive. First construct a first-order interpretation $\mathcal{M}' = (W, V')$, as the first-order interpretation associated to $\mathcal{M}$.

The interpretations of the X-predicates can be obtained as follows: Each X-predicate was obtained in the translation of some formula of form $[a]\ \psi$. Interpret $V'(X) = \{w \in W \mid \mathcal{M}, w \models [a]\ \psi\}$.

proof: ($\Leftarrow$)

If $t_{S4}(\psi, \alpha, \beta)$ is FO-satisfiable, then there exist an interpretation $\mathcal{M}' = (W, V')$, and a $w \in W$, s.t. $\mathcal{M}'[w \leftarrow \alpha] \models t_{S4}(\psi, \alpha, \beta)$.

Let $\mathcal{M} = (W, R, V)$ be the Kripke interpretation associated to $\mathcal{M}'$. Let R^* be the transitive, reflexive closure of R. Let $\mathcal{M}^* = (W, R^*, V)$.

Apply induction on ψ to prove $\mathcal{M}^*, w \models \psi$.

All cases are identical to the cases for the standard relational translation. The only exception is the $[a]$-case.

Suppose that ψ has form $[a]\varphi$.

Assume that $(w, w') \in R^*(a)$. Because R^* is the transitive closure of R, there is a chain (with possibly $n = 0$)

$$w = v_0, \ldots, v_n = w', \quad (v_0, v_1) \in V(R), \ldots, (v_{n-1}, v_n) \in V(R),$$

Since $\mathcal{M}'[w \leftarrow \alpha] \models t_{S4}([a]\varphi, \alpha, \beta)$, we have

- $\mathcal{M}'[w \leftarrow \alpha] \models X(\alpha)$,
- $\mathcal{M}' \qquad\qquad \models \forall \alpha\beta \;\; X(\alpha) \rightarrow \mathbf{R}_a(\alpha, \beta) \rightarrow X(\beta)$,
- $\mathcal{M}' \qquad\qquad \models \forall \alpha \;\; X(\alpha) \rightarrow t_{S4}(\varphi, \alpha, \beta)$.

By 'unfolding' we obtain

- $\mathcal{M}'[w' \leftarrow \alpha] \models t_{S4}(\psi, \alpha, \beta)$.

Now it is possible to apply induction and to obtain $\mathcal{M}^*, w' \models \psi$.

Regular Grammar Logics with Converse

What is the 'special' property of S4 that makes this translation work?

The closure property of S4 (reflexivity + transitivity) allows a simple (Horn clause) description of the worlds that are transitively reachable from a given world.

For S4, this description is the conjunction of formulas

$t_{S4}([a]\ \psi, \alpha, \beta) =$

$$X(\alpha),$$
$$\forall \alpha\beta \;\; X(\alpha) \rightarrow \mathbf{R}_a(\alpha, \beta) \rightarrow X(\beta),$$
$$\forall \alpha \;\; X(\alpha) \rightarrow t_{S4}(\psi, \alpha, \beta).$$

Grammar Logics

A Semi-Thue system S over alphabet Σ is a set of rules of form $u \rightarrow v$, where both $u, v \in \Sigma^*$. A Semi-Thue system is context free if for each rule $u \rightarrow v$, we have $u \in \Sigma$.

To a Semi-Thue system S belongs a rewrite relation $\Rightarrow^*$, defined as the smallest reflexive, transitive relation satisfying: If $(u \rightarrow v) \in S$, then $u_1 \cdot u \cdot u_2 \Rightarrow^* u_1 \cdot v \cdot u_2$.

For a symbol $a \in \Sigma$, the language generated by a, written as $L_S(a)$ is defined as $\{v \mid a \Rightarrow^* v\}$.

Given a Σ-model $\mathcal{M} = (W, R, V)$, the interpretations $R(a)$ can be extended to words over Σ^* as follows:

- $R(\epsilon) = \{(w, w) \mid w \in W\}$.
- $R(u \cdot v) = \{(w_1, w_3) \mid \exists w_2 \in W,$ such that $(w_1, w_2) \in R(u)$ and $(w_2, w_3) \in R(v)\}$.

A Σ-model (W, R, V) satisfies a rule $u \rightarrow v$ if $R(v) \subseteq R(u)$. A Σ-model satisfies a semi-Thue system S if it satisfies all its rules.

Example: Transitivity of a can be expressed by the rule $a \rightarrow aa$.

Reflexivity of a can be expressed by the rule $a \rightarrow \epsilon$.

S4 is characterized by $S = \{\ a \rightarrow aa, \quad a \rightarrow \epsilon\ \}$.

Converses

Definition: Let Σ be an alphabet. We call a function $\bar{\cdot}$ (of signature $\Sigma \to \Sigma$) a converse mapping if for all $a \in \Sigma$,

$$\overline{a} \neq a \text{ and } \overline{\overline{a}} = a.$$

Lemma: Given some alphabet Σ and a partition $\bar{\cdot}$, Σ can be partitioned into two disjoint sets Σ^+ and Σ^-, s.t.

- for all $a \in \Sigma^+$, $a \in \Sigma^-$,
- for all $a \in \Sigma^-$, $a \in \Sigma^+$.

Definition: A Σ-model (W, R, V) is a $(\Sigma, \bar{\cdot})$-model if it respects the converses, i.e. for each $a \in \Sigma$, $R(\overline{a}) = \{(w_1, w_2) \mid (w_2, w_1) \in R(a)\}$.

Regular Languages

A non-deterministic finite automaton (NDFA) $\mathcal{A}$ over alphabet Σ is represented by a quadruple (Q, s, F, δ), where

- Q is a finite set of states,
- $s \in Q$ is the initial state,
- $F \subseteq Q$ is the set of accepting states,
- $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation.

The relation δ can be extended to a relation $\delta^* \subseteq Q \times \Sigma^* \times Q$, which is the smallest relation satisfying:

- for all $a \in \Sigma, \quad q \in Q, \quad (q, a, q) \in \delta^*$,
- for all $a \in \Sigma, \quad u \in \Sigma^*, \quad$ if $(q_1, a, q_2) \in \delta$ and $(q_2, u, q_3) \in \delta^*$, then $(q_1, (a \cdot u), q_3) \in \delta^*$.

An NDFA $\mathcal{A} = (Q, s, F, \delta)$, accepts a word $u \in \Sigma^*$ iff there is an $f \in F$, s.t. $(s, u, f) \in \delta^*$,

For some alphabet Σ, a language L over Σ is a subset $u \subseteq \Sigma^*$.

A language L is regular if there is an NDFA $\mathcal{A}$, s.t.

$$u \in L \text{ iff } \mathcal{A} \text{ accepts } u.$$

A modal logic $\mathcal{L}$ is a regular grammar logic with converse over alphabet Σ iff there is a context-free semi-Thue system S, s.t.

1. each $L_S(a)$ is regular,
2. For a $(\Sigma, \bar{\cdot})$-model $\mathcal{M}$ holds: $\mathcal{M} = (W, R, V)$ is allowed by logic $\mathcal{L}$ iff $\mathcal{M}$ satisfies S.

Standard Modal Logics and their semi-Thue Systems

logic	$\mathrm{L}_{\mathrm{S}}(a)$	frame condition
K	$\{a\}$	(none)
KT	$\{a, \epsilon\}$	reflexivity
KB	$\{a, \overline{a}\}$	symmetry
KTB	$\{a, \overline{a}, \epsilon\}$	refl. and sym.
K4	$\{a\} \cdot \{a\}^*$	transitivity
KT4 = S4	$\{a\}^*$	refl. and trans.
KB4	$\{a, \overline{a}\} \cdot \{a, \overline{a}\}^*$	sym. and trans.
K5	$(\{\overline{a}\} \cdot \{a, \overline{a}\}^* \cdot \{a\}) \cup \{a\}$	euclideanity
KT5 = S5	$\{a, \overline{a}\}^*$	equivalence rel.
K45	$(\{\overline{a}\}^* \cdot \{a\})^*$	trans. and eucl.

Example: For modal logic S4, the language $L_S(a)$ is recognized by the automaton $\mathcal{A} = (Q, s, F, \delta)$, with

- $Q = \{X\}$.
- $s = X$,
- $F = \{X\}$,
- $\delta = \{ \ (X, a, X) \ \}$.

Relational Translation for Regular Context-Free Grammar Logics

- For a letter $a \in \Sigma^+$, we define $t_a(\alpha, \beta) = \mathbf{R}_a(\alpha, \beta)$.
- For a letter $a \in \Sigma^-$, we define $t_a(\alpha, \beta) = \mathbf{R}_a(\beta, \alpha)$.

Let $\mathcal{A} = (Q, s, F, \delta)$ be an NDFA. Let $\varphi(\alpha)$ be a *first-order* formula with one free variable α. Assume that for each state $q \in Q$, a fresh unary predicate symbol $\mathbf{q}$ is given. We define $t_{\mathcal{A}}(\alpha, \varphi)$ as the conjunction of the following formulas:

- For the initial state s, add $X_s(\alpha)$,
- for each $q \in Q$, for each $a \in \Sigma$, for each $r \in \delta(q, a)$, add

$$\forall \alpha\beta \; [\; t_a(\alpha, \beta) \rightarrow X_q(\alpha) \rightarrow X_r(\beta) \;],$$

- for each $q \in Q$, for each $r \in \delta(q, \epsilon)$, add the formula

$$\forall \alpha \; [\; X_q(\alpha) \rightarrow X_r(\alpha) \;],$$

- for each $q \in F$, add the formula

$$\forall \alpha \; [\; X_q(\alpha) \rightarrow \varphi(\alpha) \;].$$

Relational Translation for Regular Grammar Logics

- $t(p, \alpha, \beta)$ equals $\mathbf{P}(\alpha)$,
- $t(\neg p, \alpha, \beta)$ equals $\neg\mathbf{P}(\alpha)$,
- $t(\psi_1 \wedge \psi_2, \alpha, \beta)$ equals $t(\psi_1, \alpha, \beta) \wedge t(\psi_2, \alpha, \beta)$,
- $t(\psi_1 \vee \psi_2, \alpha, \beta)$ equals $t(\psi_1, \alpha, \beta) \vee t(\psi_2, \alpha, \beta)$,
- for $a \in \Sigma$, $t(\langle a \rangle\ \psi,\ \alpha, \beta)$ equals $\exists\beta\ [\ t_a(\alpha, \beta) \wedge t(\psi, \beta, \alpha)\ ]$,
- for $a \in \Sigma$, $t([a]\ \psi,\ \alpha, \beta)$ equals $t_{\mathcal{A}_a}(\alpha, t(\psi, \alpha, \beta)\ )$.

Theorem:

Let S be a regular semi-Thue system. Let φ be a modal formula. Then φ is satisfiable in a $(\Sigma, \bar{\cdot})$-model satisfying S iff $t(\psi, \alpha, \beta)$ is first-order satisfiable, where $t(\psi, \alpha, \beta)$ is the relational translation for regular grammar logics, based on the automata $\mathcal{A}_a$, for $a \in \Sigma$.

Characterizing the Borders of the Method

We want to find a logic for which the method does not work, and cannot be made to work without big modifications.

First Idea: Construct a non-regular semi-Thue system S.

This is easy: For example consider $S_i = \{a \rightarrow \overline{a}^i a\}$ for $i \geq 2$.

Problem:

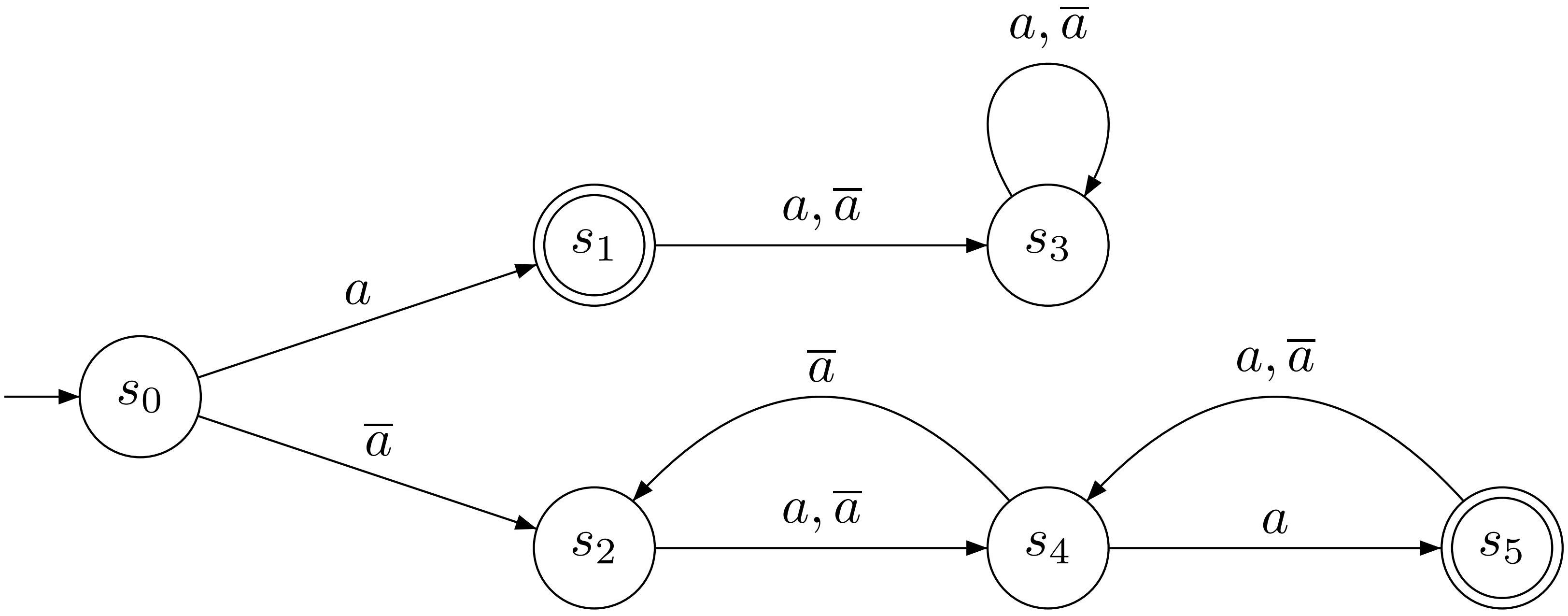
Often, there exist different semi-Thue systems which characterize the same set of Kripke interpretations.

Some other sequence of systems S_i' may characterize the same sequence of logics, but be regular.

In general, one can add rules, induced by cycles, without changing the set of $(\sigma, \bar{\cdot})$-frames.

Example: $S_2' = \{a \to \overline{a}\overline{a}a, \quad a \to \overline{a}aa\}$ characterizes the same set of frames as $S_2 = \{a \to \overline{a}\overline{a}a\}$, but it is regular.

The language $S_2' = \{a \to \overline{a}\overline{a}a, \quad a \to \overline{a}aa\}$ is accepted by the following automaton:



This can be verified by observing that it recognizes the following regular expression:

$$a \mid (\overline{a}a \mid \overline{a}\overline{a})(aa \mid a\overline{a} \mid \overline{a}a \mid \overline{a}\overline{a})^* a.$$

Behaviour of closure on Kripke models

Let Σ be an alphabet. Let v be a string over Σ. We call a string v' over Σ an expansion of v if it can be obtained from v as follows:

Let $M_v = (W, R)$ be a frame that consists of a single path, which is labeled with v. Call its starting point w_s, and its end point w_e.

Then v' must correspond to a walk from w_s to w_e in M_v, which possibly changes directions a finite number of times.

For example, aa is an expansion of itself. $aa\bar{a}\bar{a}aa$ is also an expansion of aa. Also the string $a\bar{a}aa$ is an expansion of aa.

Behaviour of closure on Kripke models (2)

Theorem: Let Σ be a semi-Thue system, which is closed under converse. Then for each pair of strings $u, v \in \Sigma^*$,

in every $(\sigma, \bar{\cdot})$-model satisfying S, $\quad R_v \subseteq R_u$

iff

there is an expansion v' of v, s.t. $u \Rightarrow_S^* v'$.

Characterization of Expansion

The set of expansions can be characterized by a semi-Thue system as follows:

Definition: Let Σ be an alphabet. We define the expansion system E_Σ of Σ as

$$\{u \cdot \overline{u} \cdot u \to u \mid u \in \Sigma^+\}.$$

Theorem Let v and v' be strings over the alphabet Σ. Then v' is an expansion of v if and only if $v' \Rightarrow^*_{E_\Sigma} v$.

Regularity of the Maximal Language

Theorem: Let S be a semi-Thue system which is closed under converse.

Let S' be the following semi-Thue system:

$$\{\ u \to v \mid \text{ there exists an expansion } v' \text{ of } v \text{ s.t. } u \Rightarrow_S^* v'\ \}.$$

Then: If S is regular, S' is also regular.

Proof: Remember the following: Let L be a language over some alphabet Σ.

Definition: Two words $w_1, w_2 \in \Sigma^*$ are equivalent (notation $w_1 \equiv w_2$) if for all $v \in \Sigma^*$, $\quad w_1 \cdot v \in L \Leftrightarrow w_2 \cdot v \in L$.

Theorem: A language L is regular iff there is an equivalence relation $\equiv$, which partitions Σ^* into a finite set of equivalence classes.

We will give an equivalence relation $\equiv$ for $\{w \mid a \Rightarrow^*_{S'} w\}$ using $\mathcal{A}_a$. Let u_1, u_2 be two strings over Σ^*.

Let M_1 be the Kripke frame which consists of a single path labeled with u_1. Call its starting world w_{s_1} and its end world w_{e_1}.

Similarly, let M_2 be the Kripke frame which consists of a single path labeled with u_2. The starting world is w_{s_2} and the end world is w_{e_2}.

Define $u_1 \equiv u_2$ iff

1. For each state q of $\mathcal{A}_a$, there exists a walk from w_{s_1} to w_{e_1} through M_1, which generates a word w_1 under which the automaton $\mathcal{A}_a$ can move from its initial state to q

   $\Leftrightarrow$

   there exists a walk from w_{s_2} to w_{e_2} through M_2, which generates a word w_2 under which the automaton $\mathcal{A}_a$ can move from its initial state to q.

2. For each two states q, q' of $\mathcal{A}_a$, there exists a walk from w_{e_1} back to w_{e_1} through M_1, which generates a word w under which the automaton $\mathcal{A}_a$ can move from state q_1 to state q_2

   $\Leftrightarrow$

   there exists a walk from w_{e_2} back to w_{e_2} through M_2, which generates a word w under which the automaton $\mathcal{A}_a$ can move from state q_1 to state q_2.

Current State of the Situation, Summary

- Many semi-Thue systems can define the same logic $\mathcal{L}$. Some may be regular, some may be not.
- If one of the semi-Thue system is regular, then the maximal system

$$S = \bigcup \{ S' \mid S \text{ is a semi-Thue system defining } \mathcal{L} \}$$

is regular.

- One can approximate $\equiv$ by checking whether $u_1 \cdot v \in L_a \Leftrightarrow u_2 \cdot v \in L_a$ by checking billions of v's.
- The implementation suggest (extremely) likey regular automata for the logics $a \rightarrow \overline{a}^i a$, but I was able to check only $i = 2$. For $i = 3$, the automaton has 19 states. For $i = 4$, the automaton has 32 states.

Conclusions, Future Work, Open Problems

- We give a simple, logspace translation from regular grammar logics with converse to the guarded fragment of first-order logic, which is EXPTIME-complete.
- For many logics, this is an optimal decision procedure.
- We have an algorithm is able to guess whether a grammar logic is a regular grammar logic. It also proposes automata. It is an absurd situation, that I don't know how to verify these automata.

Zugabe: A Resolution Decision Procedure for the Guarded Fragment

We want to know whether guarded formula F is satisfiable:

Step 1 Bring F in the form $F_1 \wedge \cdots \wedge F_n$, where each F_i has one of the following two forms

$$\forall \overline{x}_i \ [\ A_i(\overline{x}_i) \rightarrow G_i(\overline{x}_i)\ ] \text{ or } G_i.$$

The $G_i(\overline{x}_i)$ and G_i contain no essential universal quantifiers. The $G_i(\overline{x})$ contain no free variables other than $\overline{x}_i$. The G_i contain no free variables at all.

Step 2 Bring each $\forall \overline{x}_i\ [\ A_i(\overline{x}_i) \rightarrow G_i(\overline{x}_i)\ ]$ and G_i in negation normal form.

Step 3 Skolemize each $\forall \overline{x}_i\ [\ A_i(\overline{x}_i) \rightarrow G_i(\overline{x}_i)\ ]$ and G_i.

Step 4 Transform result into clauses in the usual way.

Now we have a set of clauses of the following form:

Definition: A clause is guarded if either

1. It has form $\forall \overline{x}\ \neg A \vee G_1 \vee \cdots \vee G_p$, in which A is a function free atom containing all $\overline{x}$, and all terms in the G_i are only constants, variables, or functional terms of form $f(\overline{x})$.
2. It is ground.

Resolution für Anfänger

resolution Let $c_1 = \quad A_1 \vee R_1$ and $c_2 = \quad \neg A_2 \vee R_2$ be clauses (with implicit universal quanification), s.t. A_1 and A_2 are unifiable with most general unifier Θ. Then

$$R_1\Theta \vee R_2\Theta$$

is a resolvent of c_1 and c_2. The atoms A_1, A_2 are the atoms resolved upon.

factoring Let $c = \quad A_1 \vee A_2 \vee R$ be a clause, s.t. A_1 and A_2 are unifiable with most general unifier Θ. Then

$$A_1\Theta \vee R\Theta$$

is a factor of c. The atoms A_1, A_2 are the atoms factored upon.

Resolution for Guarded Fragment

The resolution rule is restricted as follows.

1. In a guarded clause of form $\forall\overline{x}\ \neg A \vee G_1 \vee \cdots \vee G_p$ containing term of form $f(\overline{x})$, the G_i resolved upon, must contain such a term.

2. In a guarded clause of form $\forall\overline{x}\ \neg A \vee G_1 \vee \cdots \vee G_p$ without terms of form $f(\overline{x})$, only the guard can be resolved upon.

3. In a guarded clause which is ground, only literals containing a most deeply nested term can be resolved upon.

Theorem: The restricted form of resolution of the previous slide is complete for guarded clauses, and every clause that can derived by it, is also guarded.

For this reason, it is an EXPTIME decision procedure.