Mechanics of Rigid Objects

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1 Introduction

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2 Mechanics of Rigid Objects

Until now, we have simulated only point masses. A point mass is characterized by its mass, its position, and its speed. In order to simulate it, it is sufficient to compute all forces on it, and apply Newton's law, $\overline{F} = m.\overline{a}$.

Modeling objects as point masses is adequate for many applications, for example planets and performance or trajectory analysis of rockets. It is not adequate for situations where the orientation of the object is important. This is obviously the case in flight simulation, because aerodynamic and mechanic forces depend very much on the orientation of the airplane. In addition, the orientation of the plane is needed to compute the the pilot's or passenger's view.

So we need to find a way of modeling three dimensional objects. It is important to understand that three dimensional objects are nothing magical. A three dimensional object can be viewed as a collection of point masses that are connected by forces that keep the point masses in their relative position.

In reality, all three dimensional objects are somewhat elastic. If you put a force on one point of the object, at first only this point will move. After a short time, elastic forces will also move the points nearby, until, after some more time, the complete object has started moving.

This effect is usually invisible on small objects, but it is quite visible on bigger objects. For example, the wings of big airplanes distort visibly during flight through turbulence.

We define a rigid object as a collection of point masses, combined with a collection of forces that behave in such a way that they preserve the relative orientation of the point masses.

A rigid object can be simulated by primitive techniques, which we have seen already, but this will be very inefficient for big objects. Because of this, we will derive more efficient methods based on the mass distribution (inertia matrix) of the object.

Since airplanes are somewhat elastic, this model will not hold in reality. There are different solutions for this problem. First of all, one can ignore this problem. In many cases, this will still give practically useful results. Second, one can model the airplane as a small collection of rigid objects connected by springs. This has the advantage that it is more realistic, and that one can show the bending of the wings to the user. It will be still much more efficient than modeling the plane as a collection of point masses.

Since a rigid object is just a collection of point masses with some forces behaving in a special way, it inherits many properties from collections of point masses. So we start by looking at collections of point masses first.

3 Collections of Point Masses

As said before, we start by studying collections of point masses:

Definition 3.1 *We define* an object $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *as a collection of point masses. Let* $n > 0$ *be the number of point masses in the object.*

- $(m_1, \ldots, m_n)$ *are the masses.*
- $(\overline{x}_1, \ldots, \overline{x}_n)$ *are the current positions of the masses.*
- $(\overline{v}_1, \ldots, \overline{v}_n)$ *are the current speeds of the masses.*

Strictly seen, $\overline{x}$ *and* $\overline{v}$ *are functions of time, but we usually write* $\overline{x}_i$ *and* $\overline{v}_i$ *instead of* $\overline{x}_i(t)$ *and* $\overline{v}_i(t)$.

Force is a thing that always happens between two point masses. If both of the point masses are outside the object $\mathcal{O}$, then they are not our problem, and we don't care about the force. If both of the point masses are inside $\mathcal{O}$, we call the force *internal*. If one of the point masses is inside the object and the other is outside, we call the force *external*.

Definition 3.2 *Let* $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *be an object consisting of* n *point masses.*

An internal force matrix $\overline{I}$ *is an* n*-dimensional matrix of vectors, where* $\overline{I}_{i,j}$ *denotes the force that point mass* i *receives from point mass* j*. The matrix* $\overline{I}$ *has the following properties:*

1. *No point mass puts a force on itself: For all* $i,\ 1 \leq i \leq n,\ \ \overline{I}_{i,i} = (0, 0, 0)$.
2. *The force between one point mass and another point mass is always directed towards or away from the other point mass: For all* i, j *with* $i \neq j$ *and* $1 \leq i, j \leq n$*, there exists a* $\lambda \in \mathcal{R}$*, s.t.*

$$\overline{I}_{i,j} = \lambda(\overline{x}_j - \overline{x}_i).$$

3. *If one point mass feels a force from another point mass, then the other point mass feels the same force in the opposite direction: For all i, j with $1 \leq i, j \leq n$,*

$$\overline{I}_{i,j} = -\overline{I}_{j,i}.$$

An external force function $\overline{E}$ is an n-dimensional vector of vectors, where $\overline{E}_i$ denotes the external force on the i-th pointmass.

Similar to $\overline{x}$ and $\overline{v}$, the internal force matrix $\overline{I}$ and the external force vector $\overline{E}$ can change over time, but we usually don't write the dependency on t.

If we know $\overline{E}$ and $\overline{I}$, then we have all information that is required to numerically compute the behaviour of the the object. The object $\mathcal{O}$ has to fulfill the following differential equation:

$$\begin{cases} \overline{x}_i' &= \overline{v}_i \\ \overline{v}_i' &= \dfrac{\overline{E}_i + \sum_j \overline{I}_{i,j}}{m_i} \end{cases} \tag{1}$$

In order to numerically solve this system of differential equations, one can use Euler's method or a Runge-Kutta method.

3.1 Linear Properties of General Objects

We will call an object that is not rigid a *general* object. A general object can be anything from an elastic object that somewhat keeps its shape, to an object that is connected so loosely that no permanent shape can be recognized. (For example our solar system.)

Definition 3.3 *Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ be an object.*
The total mass *$m_{\mathcal{O}}$ is defined as*

$$m_{\mathcal{O}} = \sum_{i=1}^{n} m_i.$$

The center of mass *$\overline{C}_{\mathcal{O}}$ of $\mathcal{O}$ is defined as*

$$\overline{C}_{\mathcal{O}} = \frac{\sum_{i=1}^{n} m_i \overline{x}_i}{m_{\mathcal{O}}}.$$

The average speed *$\overline{V}_{\mathcal{O}}$ of $\mathcal{O}$ is defined as*

$$\overline{V}_{\mathcal{O}} = \frac{\sum_{i=1}^{n} m_i \overline{v}_i}{m_{\mathcal{O}}}.$$

Let $\overline{E}$ be an external force vector. The total force *$\overline{F}(\overline{E})$ of $\overline{E}$ on $\mathcal{O}$ is defined as*

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} \overline{E}_i.$$

If one has a general object, with external force vector $\overline{E}$ working on it, then its center of mass of moves as if it were a single point mass, with all external forces working on this single point mass:

Theorem 3.4 *Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ be an object. Let $\overline{E}$ be an external force vector. Then*

$$\overline{F}(\overline{E}) = m_{\overline{\mathcal{O}}}(\overline{V}_{\mathcal{O}})'.$$

Proof
The main insight is to see that internal forces cancel each other. This is a consequence of the fact that $I_{i,i}$ is always zero, and that for $i \neq j$, the forces are always symmetric: $\overline{I}_{i,j} = -\overline{I}_{j,i}$. It follows that:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} I_{i,j} = \overline{0}. \tag{2}$$

For each point mass, we have:

$$\overline{F}_i = m_i \overline{a}_i = m_i \overline{v}_i'.$$

$\overline{F}_i$ is the sum of internal and external forces on the i-th mass:

$$\overline{F}_i = \overline{E}_i + \sum_j \overline{I}_{i,j}.$$

Summation results in

$$\sum_{i=1}^{n} \overline{F}_i = \sum_{i=1}^{n} \overline{E}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{I}_{i,j} = \sum_{i=1}^{n} m_i \overline{v}_i'.$$

Using (2), we obtain

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{v}_i'.$$

The right hand side can be multiplied and divided by $m_{\mathcal{O}}$. In addition, one can use the fact that vectors can be differentiated component wise. The result is

$$\overline{F}(\overline{E}) = m_{\mathcal{O}} \frac{\sum_{i=1}^{n} m_i \overline{v}_i'}{m_{\mathcal{O}}} = m_{\mathcal{O}}(\overline{V}_{\mathcal{O}})'.$$

Theorem 3.4 can be used to integrate the mass center of an object $\mathcal{O}$ without knowing anything about the internal forces of $\mathcal{O}$. In a rigid object, the mass center stays fixed at one place in the object.

The center of mass is often called center of gravity. In Section 3.3, I will explain why this makes sense.

3.2 Angular Properties of General Objects: Torque and Angular Momentum

We will now define the angular counterparts of average speed and total force.

Definition 3.5 *Let $\overline{c}$ be an arbitrary position. (Called* the reference position*) Let $M = (m, \overline{x}, \overline{v})$ be a single point mass. The* angular momentum $L_M(\overline{c})$ *of M around $\overline{c}$ is defined as*

$$L_M(\overline{c}) = m(\overline{x} - \overline{c}) \times \overline{v}.$$

Let $\mathcal{O} = (\overline{m}_i, \overline{x}_i, \overline{v}_i)$ be an object. The angular momentum $\overline{L}_{\mathcal{O}}(\overline{c})$ *of $\mathcal{O}$ around $\overline{c}$ is defined as*

$$\overline{L}_{\mathcal{O}}(\overline{c}) = \sum_{i=1}^{n} m_i\, (\overline{x}_i - \overline{c}) \times \overline{v}_i.$$

Definition 3.6 *Let $\overline{F}$ be a force working on a point mass $(m, \overline{x}, \overline{v})$. Let $\overline{c}$ be some reference point. The* torque $\tau_M(\overline{F})$ *of $\overline{F}$ on M* around *(or* relative to*) $\overline{c}$, is defined as*

$$\overline{\tau}_M(\overline{F}) = (\overline{x}_i - \overline{c}) \times \overline{F}.$$

Let $\overline{E}$ be an external force vector for $\mathcal{O}$. Let $\overline{c}$ be a reference position. The torque of $\overline{E}$ around $\overline{c}$ *is defined as*

$$\overline{\tau}_{\overline{c}}(\overline{F}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i.$$

Like $\overline{E}$ itself, the torque $\overline{\tau}_c$ is time dependent, but we omit the time parameter t.

We now come to an an extremely important property, namely that the torques caused by internal forces in an object always cancel each other. We have seen already in the proof of Theorem 3.4 that internal forces always cancel each other as linear forces. We will now see that they also cancel each other as angular forces.

Theorem 3.7 *Let $\mathcal{O}$ be an object. Let $\overline{I}$ be an internal force matrix for $\mathcal{O}$. Let $\overline{c}$ be an arbitrary reference point. Let $\overline{\tau}_I$ be the total torque caused by $\overline{I}$ around $\overline{c}$, so*

$$\overline{\tau}_{\overline{c}}(\overline{I}) = \sum_{i=1}^{n}\sum_{j=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j}.$$

Then

$$\tau_{\overline{c}}(\overline{I}) = \overline{0}.$$

Proof

The essential point is the following: We already know that $i \neq j$ implies $\overline{I}_{i,j} = -\overline{I}_{i,j}$. We will show that in addition to that, also the torques caused by $\overline{I}_{i,j}$ and $\overline{I}_{j,i}$ cancel each other. This is a consequence of the fact that the internal force

between i and j is aligned along the vector $\overline{x}_j - \overline{x}_i$. Because of this alignment, there exists a real number $\lambda_{i,j}$, s.t. $\overline{I}_{i,j}$ can be written in the form $\lambda_{i,j}(\overline{x}_j - \overline{x}_i)$. We know that $\lambda_{i,i} = 0$, and that for $i \neq j$, $\lambda_{i,j} = \lambda_{j,i}$. (The $-$ disappears because $\lambda_{j,i}$ is based on $\overline{x}_i - \overline{x}_j$.)
The summation for $\overline{\tau}_I$ can be reorganized as

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j} + (\overline{x}_j - \overline{c}) \times \overline{I}_{j,i} =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} (\overline{x}_i - \overline{c}) \times \lambda_{i,j}(\overline{x}_j - \overline{x}_i) + (\overline{x}_j - \overline{c}) \times \lambda_{j,i}(\overline{x}_i - \overline{x}_j).$$

Because $\lambda_{i,j} = \lambda_{j,i}$, we can write

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} \lambda_{i,j}(\overline{x}_i \times \overline{x}_j - \overline{x}_i \times \overline{x}_i - \overline{c} \times \overline{x}_j + \overline{c} \times \overline{x}_i \ + \ \overline{x}_j \times \overline{x}_i - \overline{x}_j \times \overline{x}_j - \overline{c} \times \overline{x}_i + \overline{c} \times \overline{x}_j).$$

Because $\overline{x}_i \times \overline{x}_i = \overline{x}_j \times \overline{x}_j = \overline{0}$, and $\overline{x}_i \times \overline{x}_j = -\overline{x}_j \times \overline{x}_i$, we see that the result is $\overline{0}$. This completes the proof.

Because of the fact that internal torques always cancel each other, the total external torque on an object is equal to the total torque on all points of the object.

Theorem 3.8 *Let $\mathcal{O}$ be an object. Let $\overline{E}$ be an external force vector for $\mathcal{O}$. Let $\overline{c}$ be a reference point, and let $\overline{\tau}_{\overline{c}}(\overline{E})$ be the total torque caused by $\overline{E}$ around $\overline{c}$. For each point mass i of $\mathcal{O}$, let $\overline{F}_i$ be the total force working on m_i. Then*

$$\sum_{i=1}^{n} \overline{\tau}_{\overline{c}}(\overline{F}_i) = \overline{\tau}_{\overline{c}}(\overline{E}).$$

Proof
We have

$$\sum_{i=1}^{n} \overline{\tau}_{\overline{c}}(\overline{F}_i) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j}.$$

In the last step, we used $\overline{F}_i = \overline{E}_i + \sum_{j=1}^{n} \overline{I}_{i,j}$. Because of Theorem 3.7, this is equal to

$$\sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i = \overline{\tau}_{\overline{c}}(\overline{E}).$$

Theorem 3.9 *Let $\overline{c}$ be a position that does not change through time. Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ be a general object. Let $\overline{E}$ be the external force vector on $\mathcal{O}$, at some moment t. Let $\overline{I}$ be the internal force matrix for $\mathcal{O}$ at moment t. Then*

$$\overline{\tau}_{\overline{c}}(\overline{E}) = [\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]',$$

independent of the internal force matrix $\overline{I}$.

Proof
Newton's law holds for each point mass in $\mathcal{O}$:

$$\overline{F}_i = m_i \overline{a}_i = m_i \overline{v}_i',$$

where $\overline{F}_i$ is the total force on the i-th point mass. Left-multiplying with $\overline{x}_i - \overline{c}$ results in

$$(\overline{x}_i - \overline{c}) \times \overline{F}_i = m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i'.$$

Summation results in

$$\sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i = \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i'. \quad (3)$$

It follows from Theorem 3.8 that the left hand side is equal to $\overline{\tau}_{\overline{c}}(\overline{E})$, so that it is sufficient to show that the right hand side is equal to $[\overline{L}_{\mathcal{O}}(\overline{c})]'$. In order to show this, we compute

$$[\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]' = [\ \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i\ ]' = \sum_{i=1}^{n} m_i (\ (\overline{v}_i \times \overline{v}_i) + m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i'\ ).$$

Since $\overline{v}_i \times \overline{v}_i = \overline{0}$, we have proven that the right hand side of (3) equals $[\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]'$.

3.3 Torque relative to the Center of Gravity

The center of mass is often called the center of gravity. The reason is the fact that gravitation never causes a torque around the center of gravity.

Theorem 3.10 *Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{x})$ be a general object. Let $\overline{E}$ be an external force vector that is caused by gravity. Then $\overline{E}$ has form $(m_1\overline{g}, \ldots, m_n\overline{g})$. (All forces are in the same direction, and proportional to the mass m_i.)*
Let $\overline{c}$ be the center of mass of $\mathcal{O}$. Then $\overline{\tau}_{\overline{c}}(\overline{E}) = \overline{0}$.

Proof
Using the definition of $\overline{\tau}_{\overline{c}}(\overline{E})$, we have

$$\overline{\tau}_{\overline{c}}(\overline{E}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times m_i \overline{g}$$

Using the distributive property, this is equal to

$$\sum_{i=1}^{n} (m_i \overline{x}_i \times \overline{g}) - \sum_{i=1}^{n} (m_i \overline{c} \times \overline{g}).$$

On the left hand side, we have

$$\sum_{i=1}^{n} (m_i \overline{x}_i \times \overline{g}) = (\sum_{i=1}^{n} m_i \overline{x}_i) \times \overline{g} = m_{\mathcal{O}}\, \overline{C}_{\mathcal{O}} \times \overline{g}.$$

On the right hand side, we have

$$\sum_{i=1}^{n} (m_i \overline{c} \times \overline{g}) = (\sum_{i=1}^{n} m_i)(\overline{c} \times \overline{g}) = m_{\mathcal{O}}(\overline{c} \times \overline{g}) = m_{\mathcal{O}}(\overline{C}_O \times \overline{g}).$$

Note that theorem 3.10 can be generalized to every point on the line $\overline{C}_{\mathcal{O}} + \lambda \overline{g}$. If one assumes that 'above' and 'below' are defined by the vector $\overline{g}$, then gravity does not cause any torque in object $\mathcal{O}$ on every point that is exactly above or below the center of mass.

4 Rigid Objects

A *rigid object* is an object that is able to organize its internal forces in such a way that its point masses do not change their relative position. This means that the position of its point masses can be described by a coordinate transformation (a position and a quaternion), and that the speed of its point masses can be described by a rigid speed function. (a linear speed and an angular speed.)

4.1 Angular Speed, Angular Acceleration and Rigidity

Since the point masses in a rigid object do not change their relative positions, the speeds of the point masses can at each moment be characterized by a rigid speed function. Any other type of speed distribution would change the relative positions of the point masses. (I do not plan to prove this. If you want to prove this, you first have to define what 'relative positions' mean. You could try 'distances', but this is not enough when the object is flat. Another candidate might be 'preservation of linear combination'.)

Definition 4.1 *A* rigid speed function *has form* $\overline{V}(\overline{x}) = \overline{v}_0 + \overline{\omega} \times \overline{x}$.
An object $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *is* rigid *if there exists a rigid speed function* $\overline{V}$, *s.t. for every* i, $\overline{v}_i = \overline{V}(\overline{x}_i)$.

By differentiating the speed function, we obtain rigid acceleration.

$$\overline{A}(\overline{x}) = \overline{a}_0 + \overline{\alpha} \times \overline{x}.$$

The meaning of $\overline{V}(\overline{x})$ is: If there is a part of the rigid object on position $\overline{x}$, then $\overline{V}(\overline{x})$ is the speed of this part. Otherwise, $\overline{V}(\overline{x})$ is the speed that the part of the rigid object that is on point $\overline{x}$ would have, if it is big enough to reach $\overline{x}$.

The meaning of $\overline{A}(\overline{x})$ is similar. If a part of the rigid object reaches $\overline{x}$, then $\overline{A}(\overline{x})$ is its acceleration.
It is important to understand that in general $\overline{v}(t, \overline{x})$ and $\overline{v}(t+h, \overline{x})$ correspond to different parts of the rigid object, because the object moves in general.

4.2 Predicting the Movement of Rigid Objects

Since rigid objects also general objects, all results of section 3.1 and section 3.2 apply to rigid objects as well. This means that we already know how to compute the movements of the center of mass $\overline{C}_{\mathcal{O}}$ of an object. If we also want to know in which orientation the object is, we could do this by computing the trajectories of its point masses separately. Although this would be very inefficient, it is possible in principle. One would have to build a system of internal forces $\overline{I}$ that keep the point masses in relative position to each other, for example by simulating springs that keep the points at a fixed distance from each other. In order to obtain a realistic simulation of an airplane, one would probably need thousands of point masses, which is not feasible. Therefore, we will develope another method: Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ be a rigid object.

- Let $m_{\mathcal{O}}$ be the total mass of $\mathcal{O}$.
- Let $x_{\mathcal{O}}$ be the position of the center of mass of $\mathcal{O}$.
- Let $v_{\mathcal{O}}$ be the speed of the center of mass of $\mathcal{O}$.

We do the following: Using Definition 4.1, we derive an expression for the acceleration of each point mass. Then we will multiply this expression with the mass, so that we obtain the force on each point mass. Using this, we will compute the total torque on the rigid object as function of its angular acceleration and angular speed. Once we have this expression, we can invert it to compute angular acceleration from total torque and existing angular speed. For each point mass i in rigid object $\mathcal{O}$, we have:

$$\overline{v}_i = \overline{v}_{\mathcal{O}} + \overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}). \tag{4}$$

In order to get acceleration, we need to differentiate Equation 4. For this we use the following property:

Lemma 4.2 *Let $\overline{u}$ and $\overline{v}$ be two functions from $\mathcal{R}$ to $\mathcal{R}^3$. Then*

$$(\overline{u} \times \overline{v})' = \overline{u} \times \overline{v}' + \overline{u}' \times \overline{v}.$$

Proof
It is the usual multiplication rule, and its proof is also usual.
The result of the differentation is:

$$\overline{a}_i = \overline{a}_{\mathcal{O}} + \overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}) + \overline{\omega} \times (\overline{v}_i - \overline{v}_{\mathcal{O}}).$$

The meaning of this equation is that the linear acceleration of the i-th point mass consists of three components: **(1)** The common linear acceleration of the rigid object $\mathcal{O}$, **(2)** the acceleration resulting from change of angular velocity, and **(3)** the acceleration caused by the rotation of the object, (centripetal acceleration). By using (4) one more time, we obtain

$$\overline{a}_i = \overline{a}_{\mathcal{O}} + \overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}) + \overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}})).$$

Multiplying with the mass m_i and remembering that $\overline{F}_i = \overline{m}_i \overline{a}_i$ results in

$$\overline{F}_i = m_i \overline{a}_{\mathcal{O}} + m_i \overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}) + m_i \overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}})). \tag{5}$$

- In order to compute the linear behaviour of the rigid object, we add equation 5 for the different i. The result (using (2) in the process) is

  $$\overline{F}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{a}_{\mathcal{O}} + \sum_{i=1}^{n} m_i \overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}) + \sum_{i=1}^{n} m_i \overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}})).$$

  Since we have $\sum_{i=1}^{n} m_i = m_{\mathcal{O}}$, and $\sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) = \overline{0}$, this simplifies into

  $$\overline{F}(\overline{E}) = m_{\mathcal{O}}\, \overline{a}_{\mathcal{O}}. \tag{6}$$

  This is not surprising, because it is the same as Theorem 3.4.

- In order to establish a relation between the torque $\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E})$ around the center of mass, caused by external force vector $\overline{E}$, and angular acceleration of the rigid object, we multiply each instance of Equation 5 by the distance $(\overline{x}_i - \overline{x}_{\mathcal{O}})$. We obtain $\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{x}_i) = (\overline{x}_i - \overline{x}_{\mathcal{O}}) \times \overline{F}_i =$

  $$m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times \overline{a}_{\mathcal{O}} + m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}})) +$$
  $$m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}))).$$

  The equations (for the different i) can be added, and one can use Theorem 3.8.

  $$\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E}) = \sum_{i=1}^{n} \overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{x}_i).$$

  The result is $\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E}) =$

  $$\sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times \overline{a}_{\mathcal{O}} + \sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}})) +$$
  $$\sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}))).$$

  It is easy to check that $\sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) = (\sum_{i=1}^{n} m_i \overline{x_i}) - (\overline{x}_{\mathcal{O}} \sum_{i=1}^{n} m_i) = \overline{0}$. As a consequence, $\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E}) =$

  $$\sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}})) + \sum_{i=1}^{n} m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\omega} \times (\overline{\omega} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}))). \tag{7}$$

  The expression

  $$m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}))$$

is linear in $\overline{\alpha}$. This implies that it can be expressed by a matrix $M(\overline{\alpha})$. We call this matrix M_i, which is defined by

$$M_i(\alpha) = m_i(\overline{x}_i - \overline{x}_{\mathcal{O}}) \times (\overline{\alpha} \times (\overline{x}_i - \overline{x}_{\mathcal{O}}))$$

the *inertia matrix* of the i-th point mass. We call

$$M_{\mathcal{O}} = \sum_{i=1}^{n} M_i$$

the *inertia matrix* of the rigid object $\mathcal{O}$. Using the fact that for all vectors $\overline{v}$ and $\overline{w}$,

$$\overline{v} \times (\overline{w} \times (\overline{w} \times \overline{v})) = -\overline{w} \times (\overline{v} \times (\overline{v} \times \overline{w})),$$

(I gave a somewhat insightful proof in class, based on the fact that $\overline{\omega} \times (\overline{\omega} \times \overline{x})$ is angular acceleration for constant rotation $\overline{\omega}$ on position $\overline{x}$) and the definition of inertia matrix, (7) can be simplified into

$$\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E}) = M(\overline{\alpha}) + \overline{\omega} \times M(\overline{\omega}). \tag{8}$$

4.3 Numerical Integration of Rigid Objects

In order to compute $\overline{\alpha}$ from angular speed $\overline{\omega}$ and force vector $\overline{E}$, one can use

$$\left\{ \begin{array}{lcl} \overline{\omega}' & = & M^{-1}(\ \overline{\omega} \times M(\overline{\omega}) - \overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E})\ ) \\ q' & = & \overline{\omega}. \end{array} \right. \tag{9}$$

The linear component can be integrated with Equation 1, applied on the center of mass of the rigid object. The expression q' is not completely correct, because the quaternion is not integrated, but multiplied by the effect of $\overline{\omega}$. I have shown before how to do this.

For a single mass m on position (x_1, x_2, x_3), relative to the center of mass of the object, the inertia matrix $M_m(x_1, x_2, x_3)$ has the following form:

$$M_m(x_1, x_2, x_3) = m \begin{pmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2 + x_2^2 \end{pmatrix}$$

It is important to note that this expression, and also (9), is in external coordinates. Since computing the inertia matrix (and inverting it) can be expensive, it is better to do the main calculation in internal coordinates. It is easy to transform $\overline{\alpha}, \overline{\omega}$ and $\overline{\tau}_{\overline{x}_{\mathcal{O}}}(\overline{E})$ to internal coordinates. Once $\overline{\alpha}$ is computed, it can be transformed back to external coordinates and integrated.

Another complication is the fact that airline manufacturers do not publish the inertia matrices of their planes. So you have to guess.