Introduction to Flight Simulation (List 1)

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We try to understand orbits: Consider the differential equation

$$\begin{cases} \overline{x}' &= \overline{v} \\ \overline{v}' &= -\frac{\overline{x}}{|\overline{x}|^3} \end{cases}$$

It describes the orbit of a small object around a big object, e.g. the earth around the sun. Remember that $\overline{x}(t)$ and $\overline{v}(t)$ are functions of time.

Using two dimensional coordinates, the differential equation can be rewritten as:

$$\begin{cases} x_1' &= v_1 \\ x_2' &= v_2 \\ v_1' &= -\frac{x_1}{\sqrt{(x_1^2 + x_2^2)^3}} \\ v_2' &= -\frac{x_2}{\sqrt{(x_1^2 + x_2^2)^3}} \end{cases}$$

Remember that $x_1(t), x_2(t), v_1(t), v_2(t)$ are functions of time.

1. Prove that $x_1 v_2 - x_2 v_1$ is an invariant of the system of equations. Do you know a physical interpretation of this invariant?
2. Prove that $\frac{1}{2}(v_1^2 + v_2^2) - \frac{1}{\sqrt{x_1^2 + x_2^2}}$ is an invariant of the equations given above. Do you know a physical interpretation of this invariant?
3. (I would like to be able to prove that the orbits always lie on an ellipse, but right now, I don't know how to do that.) An ellipse with focus points $(0,0)$ and $(r,0)$ has equation:

   $$\sqrt{x_1^2 + x_2^2} + \sqrt{(x_1 - r)^2 + x_2^2} = c.$$

   It should be an invariant of solutions, but the speed is missing. Maybe it can be obtained from the first task.