Planets and Rockets

Planets

Simulating the planets is actually easy:

1. Planets can be treated as point masses.
2. Gravity is the only force that plays a role. Is this true? (At least not for the earth and the moon. Solar wind may play a role as well.)

Newton Axioms

1. Forces always act between two point masses at different positions. The direction of the force is always along the line through the two points, and the forces on the two points are opposite in direction, but equal in strength.

2. Forces between more than two point masses can be computed pairwise. (It is not completely clear to me if this is a Newton axiom. It is rather subtle.)

3. $\overline{F} = m\overline{a}$.

Gravity

Gravity is governed by the following equation:

$$F = \frac{Gm_1m_2}{r^2},$$

where

1. G is the gravitational constant, which equals

   $$G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}.$$

   (The accurracy is 1.2×10^{-4}, which means that the last two decimals are already unreliable.)

2. r is the distance between the objects.

Gravity in Vectors

Suppose that we have two point objects: The first object has mass m_1 on position $\overline{x}_1$ and the second object has mass m_2 on position $\overline{x}_2$.

Then the gravity force on the first object, caused by the second object, is defined by:

$$\overline{F} = \frac{Gm_1 m_2(\overline{x}_2 - \overline{x}_1)}{|\overline{x}_1 - \overline{x}_2|^3}.$$

Orbit Calculation

We have

$$\overline{F} = m\overline{a},$$

where

$$\overline{v} = \frac{d\overline{x}}{dt}, \text{ and } \overline{a} = \frac{d\overline{v}}{dt}.$$

Orbit Calcululation (2)

We have a group of planets with masses $m_1, \ldots, m_n$. They are on positions $\overline{x}_1(t), \ldots, \overline{x}_n(t)$. Their speeds are $\overline{v}_1(t), \ldots, \overline{v}_n(t)$.

We already know how to compute the forces: For each i with $1 \leq i \leq n$, we have

$$\overline{F}_i = \sum_{j=1}^{n} \begin{cases} j = i & (0,0,0) \\ j \neq i & \frac{G m_i m_j (\overline{x}_j - \overline{x}_i)}{|\overline{x}_j - \overline{x}_i|^3}. \end{cases}$$

For each i with $1 \leq i \leq n$, we have $\overline{a_i}(t) = \frac{\overline{F_i}(t)}{m_i}$. One can approximate:

$$\overline{v_i}(t+h) = \overline{v_i}(t) + h\overline{a_i}(t),$$

$$\overline{x_i}(t+h) = \overline{x_i}(t) + h\overline{v_i}(t).$$

The algorithm on the previous slide makes it possible to obtain accurate calculations about the solar system.

Think about this for a minute. 2000 years of human thinking can be summarized in 130 lines of C^{++} code, and checked on every cheap computer.

But the algorithm is not suitable for more complicated, derived orbits. For those, you need Runge Kutta methods. We discuss them next week.

Rocket Science

The dictionary says that
’rocket science’ = ’coś bardzo skomplikanowanego, trudnego do zrozumienia’,

but we will see that understanding rockets is much easier than understanding airplanes.

Rocket Science (2)

Suppose that we have a method of throwing away half of something at a speed of $1000m.s^{-1}$.

We start with $1000kg$.

How much mass can reach $1000m.s^{-1}$, $2000m.s^{-1}$, $11000m.s^{-1}$. (escape velocity.)

We conclude that theoretical computer science does not have a monopoly on exponential cost.

Rocket Science (3).

In reality, the throwing of mass is continuous and the complexity is even worse:

- $b(t)$ rate of fuel burn (in $kg.s^{-1}$).
- $m(t)$ mass at time t in kg.
- $\overline{v}(t)$ speed at time t in $m.s^{-1}$.
- $\overline{e}(t)$ exhaust speed in $m.s^{-1}$. (It will be constant most of the time.)

We use the law of preservation of impulse $(m.\overline{v})$. In a time interval dt, we burn $b(t).dt$ fuel. It receives a speed of $\overline{e}(t)$. This results in an added impulse of $b(t).\overline{e}(t).dt$.

The rest of the rocket has to compensate this:

$$-m(t).(\overline{v}(t+dt) - \overline{v}(t)) = b(t).\overline{e}(t).dt.$$

We divide the thing by dt, and assume that dt is very small:

$$\lim_{dt \to 0} -m(t).\frac{\overline{v}(t+dt) - \overline{v}(t)}{dt} = b(t).\overline{e}(t) \Rightarrow$$

$$-m(t).\overline{a}(t) = b(t).\overline{e}(t).$$

Since $m'(t) = b(t)$, we obtain

$$\frac{\overline{a}(t)}{\overline{e}(t)} = -\frac{m'(t)}{m(t)}.$$

We are interested in determining how mass decreases during accelaration. Suppose we let our rocket burn from t_0 to t_1. Assume that $\overline{e}(t)$ is constant.

Then

$$\int_{t_0}^{t_1} \frac{\overline{a}(t)}{\overline{e}} dt = \int_{t_0}^{t_1} -\frac{m'(t)}{m(t)}.dt \Rightarrow$$

$$\frac{\overline{v}(t_1) - \overline{v}(t_0)}{\overline{e}}.dt = \int_{t_0}^{t_1} -\frac{m'(t)}{m(t)}.dt.$$

In order to compute the second integral, we could guess that we will have exponential decay of mass, because that is what we had in the discrete case.

We have

$$\frac{d\log(m(t))}{dt} = \frac{d\log(m(t))}{dm(t)} \frac{dm(t)}{dt} = \frac{m'(t)}{m(t)}.$$

It follows that

$$\int_{t_0}^{t_1} -\frac{m'(t)}{m(t)}.dt = [-\log(m(t))]_{t_0}^{t_1} = \log(m(t_0)) - \log(m(t_1))$$

$$\log \frac{m(t_0)}{m(t_1)}.$$

This is called the Tsiolkowsky Rocket Equation.

Chain Rule

The rule that I used when differentiating $\log(m(t))$ is called the chain rule. If f and g are one-place functions, then

$$(fg)'(x) = f'(g(x)).g'(x).$$

In practice, people use the following notation, which is formally meaningless, but practically convenient:

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \frac{dg(x)}{dx} = f'(g(x)).g'(x).$$

This is the notation that I used two slides back.

Differential Equations

An equation of form $y' = F(x, y)$ is called differential equation.

A solution for the equation above is a function y of type $\mathcal{R}$ to $\mathcal{R}$, s.t. for all $x \in \mathcal{R}$,

$$y'(x) = F(y(x)).$$

For example:

$$y'(t) = c.y(t)$$

(Money on the bank, if everything goes well.)

An artificial example:

$$y'(x) = \sqrt{c.y(x)}.$$

Systems of Differential Equations

In this course, we are mostly modelling physical processes, so we assume that $y(t)$ depends on time t.

In most cases, $y(t)$ will be vector valued, i.e. of type $\mathcal{R} \rightarrow \mathcal{R}^k$ for some $n > 1$.

So, the equation gets form

$$\overline{y}(t) = \overline{F}(\ t,\ \overline{y}(t)\ ),$$

and F is of type $\mathcal{R} \times \mathcal{R}^k \rightarrow \mathcal{R}^k$.

Order of a Differential Equation

On the previous slides, the highest derivative of y that occurred in the equations, was y'. This makes the equations first-order.

In general, the order of a differential equation is defined by the highest derivative that occurs in it. E.g. $y'' = F(y, y')$ is second order.

Example:

$$y''(x) = c.\sqrt{1 + (y'(x))^2}.$$

This is the definition of a catenary, the shape that a freely hanging chain assumes.

Note that Most Grunwaldzki is not a catenary! This is because the weight per distance does not depend on the steepness.

(MG can be characterized by $y''(x) = c$.)

A higher-order differential equation can be made first-order, by increasing its dimension:

Suppose that we have the $(n+1)$-th order equation:

$$\overline{y}^{(n+1)}(t) = \overline{F}(\ \overline{y}(t), \overline{y}^{(1)}(t), \ldots, \overline{y}^{(n)}(t)\ ).$$

Define $\overline{w}_0(t) = \overline{y}(t),\ \ \overline{w}_1(t) = \overline{y}^{(1)}(t), \ldots, \overline{w}_n(t) = \overline{y}^{(n)}(t)$. Then the equation can be replaced by the system

$$\left\{ \begin{array}{lcl} \overline{w}_0'(t) & = & \overline{w}_1(t) \\ \overline{w}_1'(t) & = & \overline{w}_2(t) \\ \cdots & = & \cdots \\ \overline{w}_{n-1}'(t) & = & \overline{w}_{n-1}(t) \\ \overline{w}_n'(t) & = & \overline{F}(\ \overline{w}_0(t), \overline{w}_1(t), \ldots, \overline{w}_n(t)\ ) \end{array} \right.$$

If the original equation had dimension k, then the new equation has dimension $k.(n+1)$.

Autonous vs. Non-Autonomous

A differential equation of form $\overline{y}(t) = \overline{F}(\overline{y}(t))$ is called autonomous. If it has form $\overline{y}(t) = \overline{F}(\ t,\ \overline{y}(t)\ )$, then it is non-autonomous.

A non-autonomous equation $\overline{y}(t) = \overline{F}(\ t,\ \overline{y}(t)\ )$ can be made autonomous by adding an extra parameter x as follows:

$$\begin{cases} x' & = & 1 \\ \overline{y}'(t) & = & \overline{F}(\ x,\ \overline{y}(t)\ ) \end{cases}$$

If the old equation has dimension k, then the new equation has dimension $k+1$.

In general, differential equations can be very hard to understand. Typically, one tries to answer the following questions:

- Is there a closed form for $y(t)$?

  (In general, a differential equation can have infinitely many solutions. Some of the solutions may be closed, while others are not.)

- Is the differential equation invariant under certain operations? (For example, rotation, mirroring, reflecting, scaling).

- Do the solutions have certain invariants?

  An invariant is a function $\overline{f}$, s.t. for all solutions $\overline{y}$, for all $t_1, t_2 \in \mathcal{R}$,

$$\overline{f}(\overline{y}(t_1)) = \overline{f}(\overline{y}(t_2)).$$