Mechanics of Complex Objects

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1 Introduction

This text is part of the course ‘Flight Simulation’, WS 2014-2015.

2 Need for Rigid Objects

Until now, we have simulated only point masses. The state of a point mass is characterized by its mass, its position, and its speed. In order to simulate it, it is sufficient to compute all forces on it, and apply Newton’s law, $\overline{F} = m.\overline{a}$.

Modeling objects as point masses is adequate for many applications, for example for predicting the orbits of planets or predicting the performance and trajectory of rockets. It is not adequate for simulations where the orientation of the object matters. This is obviously the case in flight simulation, because aerodynamic and mechanic forces depend very much on the orientation of the airplane. In addition, the orientation of the plane is needed to compute the the pilot’s or passenger’s view.

So we need to find a way of modeling three dimensional objects. It is important to understand that three dimensional objects are nothing magical. A three dimensional object can be viewed as a collection of point masses that are connected by forces that keep the point masses in their relative position.

In reality, all three dimensional objects are somewhat elastic. If you put a force on one point of the object, at first only this point will move. After a short time, elastic forces will also move the points nearby, until, after some more time, the complete object has started moving.

This effect is usually invisible on small objects, but it is quite visible on bigger objects. For example, the wings of big airplanes distort visibly during flight through turbulence.

A rigid object can be viewed as a of point masses, combined with a collection of forces that behave in such a way that they preserve the relative orientation of the point masses.

A rigid object can be simulated as a group of point masses, if one models the elastic forces explicity. This is not terribly difficult to implement, but it will be very inefficient for big objects, because the number of connections between point masses is quadratic in the number of point masses. The number of connections can be reduced by only considering connections between close neighbours, but its total number will still be high.

The situation is made worse by the fact that one has to take a very small step size. In the simulation, the point masses will oscillate relative to each other. This is good in principle, because it corresponds to physical reality, but it causes instability in the numerical methods. In order to avoid this instability, one has to use a very small step size.

Because of these efficiency problems, we need to derive more efficient methods based on the mass distribution of the object.

1. Like all rigid objects, airplanes use elastic forces to keep their form. This implies that airplanes are not completely rigid.

   There are different solutions for this problem. First of all, one can ignore it. This will give practically useful results in many cases. Secondly, one can model the airplane as a small collection of rigid objects connected by springs. This has the advantage that one can show the bending of the wings to the user, and use the bending of the wings to compute aerodynamic forces more accurately. Using 3 or 5 rigid objects will be still much more efficient than modeling the aircraft as a collection of point masses.

2. Rigid objects only remain rigid if you treat them nice. This means that one have to estimate the internal forces, and check that they are within limits. If not, one has to inform the user that the plane is damaged.

3. Airplanes can have rotating parts. WW1 planes and some jet planes have engines which large rotating parts. This may cause significant gyroscopic forces, that make the handling of the plane difficult. In order to be useful, these forces have to be modeled.

   This problem can be solved by using a relaxed notion of rigid object, which allows for gyroscopic effects. This is done in [**?**], and we follow this approach.

In the rest of this text, we will derive the standard method for modeling rigid objects through its inertial matrix. First we derive some properties of collections of point masses, which are inherited by rigid objects. After that, we analyze what it actually means 'to preserve relative orientation'. Once we understand this, the resulting condition can be used to derive equations with which rigid objects (possibly enhanced with spinning parts) can be modeled.

3 Collections of Point Masses

As said before, we start by studying collections of point masses:

Definition 3.1 *We define* an object $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *as a collection of point masses. Let* $n > 0$ *be the number of point masses in the object.*

- $(m_1, \ldots, m_n)$ *are the masses.*
- $(\overline{x}_1, \ldots, \overline{x}_n)$ *are the current positions of the masses.*
- $(\overline{v}_1, \ldots, \overline{v}_n)$ *are the current speeds of the masses.*

Strictly seen, $\overline{x}$ *and* $\overline{v}$ *are functions of time, but we usually write* $\overline{x}_i$ *and* $\overline{v}_i$ *instead of* $\overline{x}_i(t)$ *and* $\overline{v}_i(t)$.

Force is a thing that always happens between two point masses. If both of the point masses are outside the object $\mathcal{O}$, then they are not our problem, and we don't care about the force. If both of the point masses are inside $\mathcal{O}$, we call the force *internal*. If one of the point masses is inside the object and the other is outside, we call the force *external*.

Definition 3.2 *Let* $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *be an object consisting of* n *point masses.*

An internal force matrix $\overline{I}$ *is an* n*-dimensional matrix of vectors, where* $\overline{I}_{i,j}$ *denotes the force that point mass* i *receives from point mass* j*. The matrix* $\overline{I}$ *has the following properties:*

1. *No point mass puts a force on itself: For all* i, $1 \leq i \leq n$, $\overline{I}_{i,i} = (0, 0, 0)$.
2. *The force between one point mass and another point mass is always directed towards or away from the other point mass: For all* i, j *with* $i \neq j$ *and* $1 \leq i, j \leq n$, *there exists a* $\lambda \in \mathcal{R}$, *s.t.*
$$\overline{I}_{i,j} = \lambda(\overline{x}_j - \overline{x}_i).$$
3. *If one point mass feels a force from another point mass, then the other point mass feels the same force in the opposite direction: For all* i, j *with* $1 \leq i, j \leq n$,
$$\overline{I}_{i,j} = -\overline{I}_{j,i}.$$

An external force function $\overline{E}$ *is an* n*-dimensional vector of vectors, where* $\overline{E}_i$ *denotes the external force on the* i*-th pointmass.*

Similar to $\overline{x}$ *and* $\overline{v}$, *the internal force matrix* $\overline{I}$ *and the external force vector* $\overline{E}$ *can change over time, but we usually don't write the dependency on* t.

If we know $\overline{E}$ and $\overline{I}$, then we have all information that is required to numerically compute the behaviour of the the object. The object $\mathcal{O}$ has to fulfill the following differential equation:

$$\left\{ \begin{array}{lcl} \overline{x}_i' & = & \overline{v}_i \\ \overline{v}_i' & = & \dfrac{\overline{E}_i + \sum_j \overline{I}_{i,j}}{m_i} \end{array} \right. \qquad (1)$$

In order to numerically solve this system of differential equations, one can use Euler's method or a Runge-Kutta method.

3.1 Linear Properties of General Objects

We will call an object that is not rigid a *general* object. A general object can be anything from an elastic object that somewhat keeps its shape, to an object that is connected so loosely that no permanent shape can be recognized. (For example our solar system.)

Definition 3.3 *Let* $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *be an object.*
The total mass $m_{\mathcal{O}}$ *is defined as*

$$m_{\mathcal{O}} = \sum_{i=1}^{n} m_i.$$

The center of mass $\overline{C}_{\mathcal{O}}$ *of* $\mathcal{O}$ *is defined as*

$$\overline{C}_{\mathcal{O}} = \frac{\sum_{i=1}^{n} m_i \overline{x}_i}{m_{\mathcal{O}}}.$$

The average speed $\overline{V}_{\mathcal{O}}$ *of* $\mathcal{O}$ *is defined as*

$$\overline{V}_{\mathcal{O}} = \frac{\sum_{i=1}^{n} m_i \overline{v}_i}{m_{\mathcal{O}}}.$$

Let $\overline{E}$ *be an external force vector. The* total force $\overline{F}(\overline{E})$ *of* $\overline{E}$ *on* $\mathcal{O}$ *is defined as*

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} \overline{E}_i.$$

If one has a general object, with external force vector $\overline{E}$ working on it, then its center of mass of moves as if it were a single point mass, with all external forces working on this single point mass:

Theorem 3.4 *Let* $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *be an object. Let* $\overline{E}$ *be an external force vector. Then*

$$\overline{F}(\overline{E}) = m_{\overline{\mathcal{O}}}(\overline{V}_{\mathcal{O}})'.$$

Proof
The main insight is to see that internal forces cancel each other. This is a consequence of the fact that $I_{i,i}$ is always zero, and that for $i \neq j$, the forces are always symmetric: $\overline{I}_{i,j} = -\overline{I}_{j,i}$. It follows that:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} I_{i,j} = \overline{0}. \tag{2}$$

For each point mass, we have:

$$\overline{F}_i = m_i \overline{a}_i = m_i \overline{v}_i'.$$

$\overline{F}_i$ is the sum of internal and external forces on the i-th mass:

$$\overline{F}_i = \overline{E}_i + \sum_j \overline{I}_{i,j}.$$

Summation results in

$$\sum_{i=1}^{n} \overline{F}_i = \sum_{i=1}^{n} \overline{E}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{I}_{i,j} = \sum_{i=1}^{n} m_i \overline{v}_i'.$$

Using (2), we obtain

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{v}_i'.$$

The right hand side can be multiplied and divided by $m_{\mathcal{O}}$. In addition, one can use the fact that vectors can be differentiated component wise. The result is

$$\overline{F}(\overline{E}) = m_{\mathcal{O}} \frac{\sum_{i=1}^{n} m_i \overline{v}_i'}{m_{\mathcal{O}}} = m_{\mathcal{O}} (\overline{V}_{\mathcal{O}})'.$$

Theorem 3.4 can be used to integrate the mass center of an object $\mathcal{O}$ without knowing anything about the internal forces of $\mathcal{O}$. The mass center responds to the forces as it were a single point mass with the total mass of object.

The center of mass is often called center of gravity. In Section 3.3, I will explain why this makes sense.

3.2 Angular Properties of General Objects: Torque and Angular Momentum

We will now define the angular counterparts of average speed and total force.

Definition 3.5 *Let $\overline{c}$ be an arbitrary position. (Called* the reference position*) Let $M = (m, \overline{x}, \overline{v})$ be a single point mass. The* angular momentum $L_M(\overline{c})$ *of M around $\overline{c}$ is defined as*

$$L_M(\overline{c}) = m(\overline{x} - \overline{c}) \times \overline{v}.$$

Let $\mathcal{O} = (\overline{m}_i, \overline{x}_i, \overline{v}_i)$ be an object. The angular momentum $\overline{L}_{\mathcal{O}}(\overline{c})$ *of $\mathcal{O}$ around $\overline{c}$ is defined as*

$$\overline{L}_{\mathcal{O}}(\overline{c}) = \sum_{i=1}^{n} m_i \, (\overline{x}_i - \overline{c}) \times \overline{v}_i.$$

Definition 3.6 *Let $\overline{F}$ be a force working on a point mass $(m, \overline{x}, \overline{v})$. Let $\overline{c}$ be some reference point. The* torque $\tau_M(\overline{F})$ *of $\overline{F}$ on M* around *(or* relative to*) $\overline{c}$, is defined as*

$$\overline{\tau}_M(\overline{F}) = (\overline{x}_i - \overline{c}) \times \overline{F}.$$

Let $\overline{E}$ *be an external force vector for* $\mathcal{O}$*. Let* $\overline{c}$ *be a reference position. The* torque of $\overline{E}$ around $\overline{c}$ *is defined as*

$$\overline{\tau}_{\overline{c}}(\overline{F}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i.$$

Like $\overline{E}$ itself, the torque $\overline{\tau}_c$ is time dependent, but we omit the time parameter t.

We now come to an an extremely important property, namely that the torques caused by internal forces in an object always cancel each other. We have seen already in the proof of Theorem 3.4 that internal forces always cancel each other as linear forces. We will now see that they also cancel each other as angular forces.

Theorem 3.7 *Let* $\mathcal{O}$ *be an object. Let* $\overline{I}$ *be an internal force matrix for* $\mathcal{O}$*. Let* $\overline{c}$ *be an arbitrary reference point. Let* $\overline{\tau}_I$ *be the total torque caused by* $\overline{I}$ *around* $\overline{c}$*, so*

$$\overline{\tau}_{\overline{c}}(\overline{I}) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j}.$$

Then

$$\tau_{\overline{c}}(\overline{I}) = \overline{0}.$$

Proof

The essential point is the following: We already know that $i \neq j$ implies $\overline{I}_{i,j} = -\overline{I}_{i,j}$. We will show that in addition to that, also the torques caused by $\overline{I}_{i,j}$ and $\overline{I}_{j,i}$ cancel each other. This is a consequence of the fact that the internal force between i and j is aligned along the vector $\overline{x}_j - \overline{x}_i$. Because of this alignment, there exists a real number $\lambda_{i,j}$, s.t. $\overline{I}_{i,j}$ can be written in the form $\lambda_{i,j}(\overline{x}_j - \overline{x}_i)$. We know that $\lambda_{i,i} = 0$, and that for $i \neq j, \quad \lambda_{i,j} = \lambda_{j,i}$. (The $-$ disappears because $\lambda_{j,i}$ is based on $\overline{x}_i - \overline{x}_j$.)

The summation for $\overline{\tau}_I$ can be reorganized as

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j} + (\overline{x}_j - \overline{c}) \times \overline{I}_{j,i} =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} (\overline{x}_i - \overline{c}) \times \lambda_{i,j}(\overline{x}_j - \overline{x}_i) + (\overline{x}_j - \overline{c}) \times \lambda_{j,i}(\overline{x}_i - \overline{x}_j).$$

Because $\lambda_{i,j} = \lambda_{j,i}$, we can write

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} \lambda_{i,j}(\overline{x}_i \times \overline{x}_j - \overline{x}_i \times \overline{x}_i - \overline{c} \times \overline{x}_j + \overline{c} \times \overline{x}_i \ + \ \overline{x}_j \times \overline{x}_i - \overline{x}_j \times \overline{x}_j - \overline{c} \times \overline{x}_i + \overline{c} \times \overline{x}_j).$$

Because $\overline{x}_i \times \overline{x}_i = \overline{x}_j \times \overline{x}_j = \overline{0}$, and $\overline{x}_i \times \overline{x}_j = -\overline{x}_j \times \overline{x}_i$, we see that the result is $\overline{0}$. This completes the proof.

Because of the fact that internal torques always cancel each other, the total external torque on an object is equal to the total torque on all points of the object.

Theorem 3.8 *Let $\mathcal{O}$ be an object. Let $\overline{E}$ be an external force vector for $\mathcal{O}$. Let $\overline{c}$ be a reference point, and let $\overline{\tau}_{\overline{c}}(\overline{E})$ be the total torque caused by $\overline{E}$ around $\overline{c}$. For each point mass i of $\mathcal{O}$, let $\overline{F}_i$ be the total force working on m_i. Then*

$$\sum_{i=1}^{n} \overline{\tau}_{\overline{c}}(\overline{F}_i) = \overline{\tau}_{\overline{c}}(\overline{E}).$$

Proof
We have

$$\sum_{i=1}^{n} \overline{\tau}_{\overline{c}}(\overline{F}_i) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{I}_{i,j}.$$

In the last step, we used $\overline{F}_i = \overline{E}_i + \sum_{j=1}^{n} \overline{I}_{i,j}$. Because of Theorem 3.7, this is equal to

$$\sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{E}_i = \overline{\tau}_{\overline{c}}(\overline{E}).$$

Theorem 3.9 *Let $\overline{c}$ be a position that does not change through time. Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ be a general object. Let $\overline{E}$ be the external force vector on $\mathcal{O}$, at some moment t. Let $\overline{I}$ be the internal force matrix for $\mathcal{O}$ at moment t. Then*

$$\overline{\tau}_{\overline{c}}(\overline{E}) = [\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]',$$

independent of the internal force matrix $\overline{I}$.

Proof
Newton's law holds for each point mass in $\mathcal{O}$:

$$\overline{F}_i = m_i \overline{a}_i = m_i \overline{v}_i',$$

where $\overline{F}_i$ is the total force on the i-th point mass. Left-multiplying with $\overline{x}_i - \overline{c}$ results in

$$(\overline{x}_i - \overline{c}) \times \overline{F}_i = m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i'.$$

Summation results in

$$\sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times \overline{F}_i = \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i'. \tag{3}$$

It follows from Theorem 3.8 that the left hand side is equal to $\overline{\tau}_{\overline{c}}(\overline{E})$, so that it is sufficient to show that the right hand side is equal to $[\overline{L}_{\mathcal{O}}(\overline{c})]'$. In order to show this, we compute

$$[\ \overline{L}_{\mathcal{O}}(\overline{c})\ ]' = [\ \sum_{i=1}^{n} m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i\ ]' = \sum_{i=1}^{n} m_i (\ (\overline{v}_i \times \overline{v}_i) + m_i (\overline{x}_i - \overline{c}) \times \overline{v}_i'\ ).$$

Since $\overline{v}_i \times \overline{v}_i = \overline{0}$, we have proven that the right hand side of (3) equals $[\, \overline{L}_{\mathcal{O}}(\overline{c})\,]'$.

It follows from Theorem 3.9 that any group of point masses on which no external force works, preserves angular momentum around every non-moving position.

3.3 Torque relative to the Center of Gravity

The center of mass is often called the center of gravity. The reason is the fact that gravitation never causes a torque around the center of mass.

Theorem 3.10 *Let $\mathcal{O} = (\overline{m}, \overline{x}, \overline{x})$ be a general object. Let $\overline{E}$ be an external force vector that is caused by gravity. Then $\overline{E}$ has form $(m_1\overline{g}, \ldots, m_n\overline{g})$. (All forces are in the same direction, and proportional to the mass m_i.)*
Let $\overline{c}$ be the center of mass of $\mathcal{O}$. Then $\overline{\tau}_{\overline{c}}(\overline{E}) = \overline{0}$.

Proof
Using the definition of $\overline{\tau}_{\overline{c}}(\overline{E})$, we have

$$\overline{\tau}_{\overline{c}}(\overline{E}) = \sum_{i=1}^{n} (\overline{x}_i - \overline{c}) \times m_i\overline{g}$$

Using the distributive property, this is equal to

$$\sum_{i=1}^{n} (m_i\overline{x}_i \times \overline{g}) - \sum_{i=1}^{n} (m_i\overline{c} \times \overline{g}).$$

On the left hand side, we have

$$\sum_{i=1}^{n} (m_i\overline{x}_i \times \overline{g}) = (\sum_{i=1}^{n} m_i\overline{x}_i) \times \overline{g} = m_{\mathcal{O}}\, \overline{C}_{\mathcal{O}} \times \overline{g}.$$

On the right hand side, we have

$$\sum_{i=1}^{n} (m_i\overline{c} \times \overline{g}) = (\sum_{i=1}^{n} m_i)(\overline{c} \times \overline{g}) = m_{\mathcal{O}}(\overline{c} \times \overline{g}) = m_{\mathcal{O}}(\overline{C}_O \times \overline{g}).$$

Note that theorem 3.10 can be generalized to every point on the line $\overline{C}_{\mathcal{O}} + \lambda\overline{g}$. If one assumes that 'above' and 'below' are defined by the vector $\overline{g}$, then gravity does not cause any torque in object $\mathcal{O}$ on every point that is exactly above or below the center of mass.

4 Modelling Rigid Objects

A *rigid object* is an object that is able to organize its internal forces in such a way that its point masses do not change their relative positions. This means that the positions of the point masses can be described by a coordinate transformation (a single position and a quaternion). and that the speed of its point masses can be described by a rigid speed function. (a single linear speed and an angular speed.)

Rigid objects do not exist in physical reality, but close approximations are all around us. Physical objects slightly change form, when a force is applied on them. This change in form causes elastic forces, which more or less manage to preserve the original shape of the object.

4.1 Angular Speed and Rigidity

Since the point masses in a rigid object cannot not change their relative positions, their relative speeds and relative positions are strongly restricted.

Definition 4.1 *A* rigid speed function *has form* $\overline{V}(\overline{x}) = \overline{v} + \overline{\omega} \times \overline{x}$.
An object $\mathcal{O} = (\overline{m}, \overline{x}, \overline{v})$ *is* rigid *(at time t) if there exists a rigid speed function* $\overline{V}$*, s.t. for each of its point masses* $(m_i, \overline{x}_i, \overline{v}_i)$, $\quad \overline{v}_i = \overline{V}(\overline{x}_i)$.

$\overline{v}$ and $\overline{\omega}$ are parameters that may change over time. The intuitive meaning of the rigid speed function is tricky: At moment t, if there happens to be a point mass at position $\overline{x}$, it moves with speed $\overline{v}(t) + \omega(t) \times \overline{x}$.

The parameter $\overline{v}$ is not the speed of the rigid object, but the speed that a point mass present at position $(0, 0, 0)$ would have.

The acceleration of a point mass at position $\overline{x}$ (at time t) cannot be obtained by simply differentiating the speed function. Assume that at time t a point mass is at position $\overline{x}$. Let h be a small amount of time. At time $t + h$, the speed at position $\overline{x}$ will be approximately $\overline{v}(t) + h.\overline{v}'(t) + (\ \overline{\omega}(t) + h.\overline{\omega}'(t)\ ) \times \overline{x}$.

This is not the speed of the point mass because the point mass has moved from $\overline{x}$ to $\overline{x} + h.\overline{V}_t(\overline{x}) = \overline{x} + h.\overline{v}(t) + h.\overline{\omega}(t) \times \overline{x}$.

In order to get the new speed of the point mass, the new speed function has to be applied on the new position. The result is:

$$\overline{v}(t) + h.\overline{v}'(t) + (\ \overline{\omega}(t) + h.\overline{\omega}'(t)\ ) \times (\ \overline{x} + h.\overline{v}(t) + h.\overline{\omega}(t) \times \overline{x}\ ).$$

We will write $\overline{W}(t+h)$ for this expression in order to distinguish it from $\overline{V}_t(\overline{x})$. Our aim is to find the acceleration of the point mass, so we want to determine the value of

$$\overline{a}(t) = \lim_{h \to 0} \frac{\overline{W}(t+h) - \overline{V}_t(\overline{x})}{h}.$$

In $\overline{W}(t + h)$, we can apply the distributive law on $\times$, and remove all terms that are quadratic in h, becasue they will not contribute to the limit. We get $\overline{W}(t+h) =$

$$\overline{v}(t) + h.\overline{v}'(t) + \overline{\omega}(t) \times \overline{x} + h.\overline{\omega}(t) \times \overline{v}(t) + h.\overline{\omega}(t) \times (\ \overline{\omega}(t) \times \overline{x}\ ) + h.\overline{\omega}'(t) \times \overline{x}.$$

$$=$$

$$\overline{v}(t) + h.\overline{v}'(t) + (\ \overline{\omega}(t) + h.\overline{\omega}'(t)\ ) \times \overline{x} + h.\overline{\omega}(t) \times \overline{v}(t) + h.\overline{\omega}(t) \times (\ \overline{\omega}(t) \times \overline{x}\ ).$$

We see that

$$\overline{a}(t) = \overline{v}'(t) + \overline{\omega}'(t) \times \overline{x} + \overline{\omega}(t) \times (\ \overline{v}(t) + \overline{\omega}(t) \times \overline{x}\ ). \tag{4}$$

4.2 Predicting the Trajectories of Rigid Objects

Since rigid objects are still general objects, all results of Sections 3.1 and 3.2 apply to rigid objects. This means that we already know how to compute the movement of the center of mass of a rigid object. It can be predicted by Theorem 3.4.

We can also predict, for every reference point $\overline{c}$, how the angular momentum around this point will change under a known torque around $\overline{c}$. This is predicted by Theorem 3.9.

The only thing that is missing at this point, is a connection between the change in rigid speed function and the change in momentum. Once we have this connection, we can use it to obtain an explicit expression for change in rigid speed function.

In order to connect change in the rigid speed function to change in momentum and speed, we use Equation 4, which can be applied to every point mass of the rigid object.

$$\overline{a}_i = \overline{v}' + \overline{\omega}' \times \overline{x}_i + \overline{\omega} \times (\ \overline{v} + \overline{\omega} \times \overline{x}_i \ ).$$

All variables are functions in the time t, but we omitted the time argument, because we will be only considering a single time moment in the rest of this section.

Multiplying with the mass m_i and remembering that $\overline{F}_i = m_i \overline{a}_i$ results in

$$\overline{F}_i = m_i.\overline{v}' + m_i.\overline{\omega}' \times \overline{x}_i + m_i.\overline{\omega} \times (\ \overline{v} + \overline{\omega} \times \overline{x}_i \ ). \tag{5}$$

- Since we are interested in connecting torque to momentum, we left-multiply (5) with the position $\overline{x}_i$.

  $$\overline{x}_i \times \overline{F}_i = m_i.\overline{x}_i \times \overline{v}' + m_i.\overline{x}_i \times (\overline{\omega}' \times \overline{x}_i) + m_i.\overline{x}_i \times (\overline{\omega} \times (\ \overline{v} + \overline{\omega} \times \overline{x}_i \ )).$$

  At this point, the equations (for the different point masses) can be summed, and one can use Theorem 3.8 to eliminate the internal torques. The result is:

  $$\overline{\tau}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{x}_i \times \overline{v}' + \sum_{i=1}^{n} m_i \overline{x}_i \times (\overline{\omega}' \times \overline{x}_i) + \sum_{i=1}^{n} m_i \overline{x}_i \times (\overline{\omega} \times (\ \overline{v} + \omega \times \overline{x}_i \ )).$$

  Since $\sum_{i=1}^{n} m_i \overline{x}_i = m_{\mathcal{O}} \overline{C}_{\mathcal{O}}$, we can write $\overline{\tau}(E) =$

  $$m_{\mathcal{O}} \overline{C}_{\mathcal{O}} \times \overline{v}' + \sum_{i=1}^{n} m_i \overline{x}_i \times (\overline{\omega}' \times \overline{x}_i) + m_{\mathcal{O}} \overline{C}_{\mathcal{O}} \times (\overline{\omega} \times \overline{v}) + \sum_{i=1}^{n} m_i \overline{x}_i \times (\overline{\omega} \times (\overline{\omega} \times \overline{x}_i)).$$

  The expression

  $$m_i \overline{x}_i \times (\overline{\omega}' \times \overline{x}_i)$$

  is linear in $\overline{\omega}'$. This implies that it can be written in the form $I_i(\overline{\omega}')$, where I_i is a matrix. The matrix I_i is called *the inertia matrix* (or *inertial matrix*) of the point mass m_i at position $\overline{x}_i$. We will give the precise form in Section 4.3. The sum $I_{\mathcal{O}}$ of the matrices $\sum_{i=1}^{n} I_i$ is called the *inertia matrix* of the object.

Using the fact that for all vectors $\overline{v}$ and $\overline{w}$,

$$\overline{v} \times (\overline{w} \times (\overline{w} \times \overline{v})) = -\overline{w} \times (\overline{v} \times (\overline{v} \times \overline{w})),$$

and the definition of inertia matrix, the expression can be simplified into

$$\overline{\tau}(\overline{E}) = m_{\mathcal{O}} \overline{C}_{\mathcal{O}} \times (\overline{v}' + \overline{\omega} \times \overline{v}) + I_{\mathcal{O}}(\omega') + \overline{\omega} \times I_{\mathcal{O}}(\overline{\omega}). \tag{6}$$

- In order to get an expression for the linear behavior of the rigid object, we start with (5) again:

$$\overline{F}(\overline{E}) = \sum_{i=1}^{n} m_i \overline{v}' + \sum_{i=1}^{n} m_i \overline{\omega}' \times \overline{x}_i + \sum_{i=1}^{n} m_i \overline{\omega} \times (\overline{v} + \overline{\omega} \times \overline{x}_i).$$

Using $\sum_{i=1}^{n} m_i = m_{\mathcal{O}}$, and $\sum_{i=1}^{n} m_i \overline{x}_i = m_{\mathcal{O}} \overline{C}_{\mathcal{O}}$, we get

$$\overline{F}(\overline{E}) = m_{\mathcal{O}}(\ \overline{v}' + \overline{\omega}' \times \overline{C}_{\mathcal{O}} + \overline{\omega} \times (\overline{v} + \overline{\omega} \times \overline{C}_{\mathcal{O}})\ ). \tag{7}$$

4.3 Numerical Integration of Rigid Objects

Equations (6) and (7) are almost usable for the modelling of rigid objects, but some adaptations are necessary, that we will now describe. When we start computing the derivatives $\overline{\omega}'$ and $\overline{v}'$, the values of $\overline{v}, \overline{\omega}, m_{\mathcal{O}}, I_{\mathcal{O}}$, and $\overline{C}_{\mathcal{O}}$ are known. The linear force $\overline{F}(\overline{E})$ and the torque $\overline{\tau}(\overline{E})$ have to be determined from the state of the object, i.e. from $(q, \overline{x}, \overline{v}, \overline{\omega})$, and possible other state variables. This leaves us with two equations containing two unknowns, so that in principle $\overline{v}'$ and $\overline{\omega}'$ can be determined.

It is possible to find $\overline{v}'$ and $\overline{\omega}'$ using Gauss elimination, but it is much easier to switch to a coordinate system with origin in $\overline{C}_{\mathcal{O}}$. In that case, we have $\overline{C}_{\mathcal{O}} = (0,0,0)$, so that the equations can be replaced by

$$\begin{cases} \overline{\tau}(\overline{E}) = I_{\mathcal{O}}(\overline{\omega}') + \overline{\omega} \times I_{\mathcal{O}}(\overline{\omega}) \\ \overline{F}(\overline{E}) = m_{\mathcal{O}}(\ \overline{v}' + \overline{\omega} \times \overline{v}\ ) \end{cases}$$

Now it is easy to obtain

$$\begin{cases} \overline{\omega}' &= I_{\mathcal{O}}^{-1}(\ \overline{\tau}(\overline{E}) - \overline{\omega} \times I_{\mathcal{O}}(\omega)\ ) \\ \overline{v}' &= \dfrac{\overline{F}(\overline{E})}{m_{\mathcal{O}}} - \overline{\omega} \times \overline{v} \end{cases} \tag{8}$$

The reason why this equation is different from Theorem 3.4 is the fact that $\overline{v}'$ does not denote acceleration of the mass center of the rigid object, but the acceleration of the rigid speed function at $(0,0,0)$. This is the speed of the part of the rigid object that happens to be present at $(0,0,0)$.

Equations 8 have two problems:

1. It is a problem to represent a moving object by a rigid speed function, using a fixed point of reference. This is the same problem that we already saw in two dimensions. At a given moment t, one defines the rigid speed function using the linear speed at the mass center, and the angular speed of the object. At $t + h$, integration would produce a rigid speed function that still uses the linear speed at the mass center of time t.

   In order to continue using the position of the mass center, the rigid speed function has to be corrected at every iteration. The effect of this correction will be that the term $\overline{\omega} \times \overline{v}$ disappears.

2. The inertial matrix depends on the orientation of the object. Recomputing the inertial matrix at each iteration of the integration procedure is possible, but inefficient. This problem can be solved by making the coordinate system not only move, but also rotate with the object.

The complete procedure is:

1. Fix a coordinate system at the mass center of the object.

2. Collect all forces and their acting points. Compute total force and total torque, relative to the current position of the mass center, and orientation of the object.

3. Compute angular acceleration, using Equations 8.

4. Compute new position, new orientation, new rigid speed function at time $t + h$.

5. Correct the rigid speed function for the new position of mass center.

Let $\overline{F}$ be the total force, let $\overline{T}$ be the total torque, expressed in the coordinate system that is defined by the current position and orientation of the object.

$$\begin{cases} \overline{\omega}_{t+h} = \overline{\omega}_t + I^{-1}(\ \overline{T} - \overline{\omega}_t \times I(\overline{\omega}_t)\ ) \\ \overline{V}_{t+h} = \overline{V}_t + h(\ \frac{\overline{F}}{M} \boxed{-\overline{\omega}_t \times \overline{V}_t}\ ) \\ q_{t+h} = (\|\overline{\omega}_t\| \cos \frac{1}{2} \|\overline{\omega}_t\| h;\ \overline{\omega}_t \sin \frac{1}{2} \|\overline{\omega}_t\| h).q_t \\ \overline{X}_{t+h} = \overline{X}_t + h.\overline{V}_t \end{cases}$$

The term $\overline{\omega}_t \times \overline{V}_t$ is boxed, because it disappears when the reference position is changed. The quaternion $(\|\overline{\omega}_t\| \cos \frac{1}{2} \|\overline{\omega}_t\| h;\ \overline{\omega}_t \sin \frac{1}{2} \|\overline{\omega}_t\| h)$ is the effect of angular speed $\overline{\omega}_t$ over a time h.

The force $\overline{F}$ and torque $\overline{T}$ must be expressed in internal coordinates of the object.

Note that, altough the inertial matrix usually doesn't change much, it still may change in certain situations: Rockets burn their fuel very quickly, military planes and fire extinguishing planes may drop half of their mass in a few seconds. Even passenger planes may burn one third of their mass on a long flight.