Introduction to Flight Simulation (List 2)

Deadlie: 29.10.2012

1. Consider the following Butcher tableau:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ 1 & \frac{1}{4} & \frac{3}{4} & & \\ 1 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & \\ \hline & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{array}$$

   (It is just an example. It is not supposed to give a good order.)

   (a) Write out the complete method, using k_1, k_2, k_3, k_4 and assuming function $\overline{F}$.

   (b) On the slides, I assumed that the function $\overline{F}$ is autonomous. The standard, non-autonomous definition can be obtained by putting $\overline{F} = \lambda x : \mathcal{R} \;\; \overline{y} : \mathcal{R}^n \;\; (1, \overline{G}(x, \overline{y}) \;)$. Using this fact, give the standard, non-autonomous definition of the RK method, defined by the tableau above.

   (c) Do the same with the standard Runge Kutta method RK41. The result is the standard Runge-Kutta method that you usually find in text books.

2. Consider the differential equation

$$\overline{a} = \frac{\overline{x}}{\|\overline{x}\|} F(\|\overline{x}\|). \tag{1}$$

   It models the trajectory of a point mass, on which works a force that always points towards or away from the origin. The strength of the force depends only on the distance $\|\overline{x}\|$ from the center.

   (a) Write Equation 1 as a real valued, first-order system of equations.

   (b) Prove that $\overline{x} \times \overline{v}$ is an invariant of Equation 1. You can do this by writing out the definition of $\overline{x} \times \overline{v}$. If you are brave enough, you can do it directly in vector calculus.

(c) Let $E(x)$ be a function for which $E'(x) = F(x)$. One may assume that $E(x)$ is *potential energy*. Show that $\frac{1}{2}\|\overline{v}\|^2 - E(\|\overline{x}\|)$ is an invariant of Equation 1.

You now have shown *preservation of energy* for general force distribution.

3. We now take $F(\overline{x}) = -\dfrac{1}{\|\overline{x}\|^2}$. Prove that the vector $\overline{v} \times (\overline{x} \times \overline{v}) - \frac{\overline{x}}{\|\overline{x}\|}$, which can be written in the form

$$\begin{pmatrix} v_2(x_1v_2 - x_2v_1) - \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ -v_1(x_1v_2 - x_2v_1) - \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}} \end{pmatrix}$$

is now an invariant of Equation 1. It is called the 'Laplace-Runge-Lenz vector'.

This is not part of the exercise: Since the LRL-vector is an invariant of Equation 1, (if we assume that F is defined as in Task 3), it always points in the same direction and always has the same length. Let us call the length of the LRL-vector R. At a given point in time, let φ be the angle between the LRL-vector and the position of the point mass.

We calculate the dot product of the position $\overline{x}$ and the LRL-vector. It is equal to $x.R.\cos\varphi$, where x is the length of the position vector. Using the definition of the LRL-vector, we have:

$$x.R.\cos\varphi = \overline{x}.(\ \overline{v} \times (\overline{x} \times \overline{v}) - \frac{\overline{x}}{\|\overline{x}\|}\ ) = \overline{x}.(\overline{v} \times (\overline{x} \times \overline{v})) - \frac{\overline{x}.\overline{x}}{\|\overline{x}\|}.$$

We know that $\overline{x} \times \overline{v}$ is an invariant of the system. (You proved this in task 2b.) Define $\overline{\alpha} = \overline{x} \times \overline{v}$. (for *angular momentum*) We have $\overline{x}.(\overline{v} \times \overline{\alpha}) = (\overline{x} \times \overline{v}).\overline{\alpha} = \overline{\alpha}.\overline{\alpha} = \alpha^2$. Since we also have $\frac{\overline{x}.\overline{x}}{\|\overline{x}\|} = \|\overline{x}\| = x$, the right hand side can be simplified to $\alpha^2 - x$, so that we obtain: $x.R.\cos\varphi = \alpha^2 - x$. If we isolate x, we see that

$$x = \frac{\alpha^2}{1 + R.\cos\varphi}.$$

Dependent on the value of R, this either characterizes an *ellipse* ($|R| < 1$), a *parabole* ($|R| = 1$), or a *hyperbole* ($|R| > 1$).

The fact that the planets move around in ellipses was empirically observed for Mars by Johannes Kepler in 1605 (and postulated for the other planets), and proven from first principles by Isaac Newton in 1684.