Planets and Rockets

Planets

Simulating planets is surprisingly easy:

1. Planets can be treated as point masses.
2. Gravity is the only force that plays a role. Is this true? (At least not for the earth and the moon. Solar wind may play a role as well.)

Newton Axioms

1. Forces always act between two point masses at different positions. The direction of the force is always along the line through the two points, and the forces on the two points are opposite in direction, but equal in strength.

2. Forces between more than two point masses can be computed pairwise. (It is not completely clear to me if this is a Newton axiom. It is rather subtle.)

3. For individual point masses, we have:

$$\overline{F} = m\overline{a}.$$

Gravity

Gravity is governed by the following equation:

$$F = \frac{Gm_1m_2}{r^2},$$

where

1. G is the gravitational constant, which equals

   $$G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}.$$

   (The accurracy is 1.2×10^{-4}, which means that the last two decimals are already unreliable.)

2. r is the distance between the objects.

Gravity in Vectors

Suppose that we have two point objects: The first object has mass m_1 on position $\overline{x}_1$ and the second object has mass m_2 on position $\overline{x}_2$.

Then the gravity force on the first object, caused by the second object, is defined by:

$$\overline{F} = \frac{G m_1 m_2 (\overline{x}_2 - \overline{x}_1)}{|\overline{x}_1 - \overline{x}_2|^3}.$$

Orbit Calculation

We have

$$\overline{F} = m\overline{a},$$

where

$$\overline{v} = \frac{d\overline{x}}{dt}, \text{ and } \overline{a} = \frac{d\overline{v}}{dt}.$$

Orbit Calcululation (2)

We have a group of planets with masses $m_1, \ldots, m_n$. They are on positions $\overline{x}_1(t), \ldots, \overline{x}_n(t)$. Their speeds are $\overline{v}_1(t), \ldots, \overline{v}_n(t)$.

We already know how to compute the forces: For each i with $1 \leq i \leq n$, we have

$$\overline{F}_i = \sum_{j=1}^{n} \begin{cases} j = i & (0,0,0) \\ j \neq i & \frac{G m_i m_j (\overline{x}_j - \overline{x}_i)}{|\overline{x}_j - \overline{x}_i|^3} \end{cases}.$$

For each i with $1 \leq i \leq n$, we have $\overline{a_i}(t) = \frac{\overline{F_i}(t)}{m_i}$. One can approximate:

$$\begin{cases} \overline{v_i}(t+h) = \overline{v_i}(t) + h\overline{a_i}(t), \\ \overline{x_i}(t+h) = \overline{x_i}(t) + h\overline{v_i}(t). \end{cases}$$

The algorithm on the previous slide makes it possible to obtain accurate calculations about the solar system.

Think about this for a minute. 2000 years of human thinking can be summarized in 130 lines of C^{++} code, and checked on every cheap computer.

But the algorithm is not suitable for more complicated, derived orbits. For those, you need Runge Kutta methods. We discuss them next week.

Rocket Science

The dictionary says that
’rocket science’ = ’coś bardzo skomplikanowanego, trudnego do zrozumienia’,

but we will see that understanding rocket trajectories is not difficult at all: It is much easier than understanding airplanes.

Rocket Science (2)

Suppose that we have a method of throwing away half of something at a speed of $1000 m.s^{-1}$.

We start with $1000 kg$.

How much mass can reach $1000 m.s^{-1}$, $2000 m.s^{-1}$, $11000 m.s^{-1}$? (escape velocity.)

(Theoretical computer science does not have a monopoly on exponential cost. Moreover, rocket scientists can prove exponential cost, where computer scientists can only conjecture.)

Rocket Science (3).

In reality, the throwing of mass is continuous and the complexity is even worse:

- $b(t)$ rate of fuel burn (in $kg.s^{-1}$).
- $m(t)$ mass at time t in kg.
- $\overline{v}(t)$ speed at time t in $m.s^{-1}$.
- $\overline{e}(t)$ exhaust speed in $m.s^{-1}$. (It will be constant most of the time.)

We use the law of preservation of impulse ($m.\overline{v}$). In a time interval h, we burn $h.b(t)$ fuel. The fuel is thrown away with a speed of $\overline{e}(t)$. This results in an added impulse of $h.b(t).\overline{e}(t)$.

The rest of the rocket has to compensate this:

$$-m(t).(\overline{v}(t+h) - \overline{v}(t)) = h.b(t).\overline{e}(t).$$

We divide by h, and assume that h is very small:

$$\lim_{h \to 0} -m(t).\frac{\overline{v}(t+h) - \overline{v}(t)}{h} = b(t).\overline{e}(t) \Rightarrow$$

$$-m(t).\overline{a}(t) = b(t).\overline{e}(t).$$

Since $m'(t) = b(t)$, we obtain

$$\overline{a}(t) = -\frac{m'(t)}{m(t)}.\overline{e}(t)$$

We want to know how much fuel should be burnt in order to obtain a certain change in speed. We assume that we burn fuel from t_0 to t_1 at a rate $\overline{e}(t)$. Then

$$\int_{t_0}^{t_1} \overline{a}(t).dt = -\overline{e} \int_{t_0}^{t_1} \frac{m'(t)}{m(t)}.dt \Rightarrow$$

$$\overline{v}(t_1) - \overline{v}(t_0) = -\overline{e} \int_{t_0}^{t_1} -\frac{m'(t)}{m(t)}.dt.$$

In order to compute the second integral, one could guess that we will have exponential decay of mass, because that is what we had in the discrete case.

We have

$$\frac{d\log(m(t))}{dt} = \frac{d\log(m(t))}{dm(t)} \frac{dm(t)}{dt} = \frac{m'(t)}{m(t)}.$$

It follows that

$$-\overline{e}\int_{t_0}^{t_1}\frac{m'(t)}{m(t)}.dt = -\overline{e}.(\ \log m(t_1) - \log m(t_0)\ ) =$$

$$\overline{e}.\log\frac{m(t_0)}{m(t_1)}.$$

The result is the Tsiolkowsky (Ciółkowski) Rocket Equation:

$$\Delta\overline{v} = \overline{e}.\log\frac{m(t_0)}{m(t_1)}.$$

Chain Rule

The rule that I used when differentiating $\log(m(t))$ is called the chain rule. If f and g are one-place functions, then

$$(fg)'(x) = f'(g(x)).g'(x).$$

In practice, people use the following notation, which is formally meaningless, but practically convenient:

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \frac{dg(x)}{dx} = f'(g(x)).g'(x).$$

This is the notation that I used two slides back.

The effective exhaust velocity $\overline{e}$ can be 3500 $m.s^{-1}$ for a liquid oxygen/kerosene engine, and 4500 $m.s^{-1}$ for a liquid oxygen/liquid hydrogen mixture.

The Space Shuttle weighed 74 745 kg empty. It had a payload of 29 445 kg.

Its external tank had an empty weight of 35 379 kg and contained 615 627 kg of liquid O_2, and 102 378 kg of liquid H_2, which was burned in 8 minutes.

Its two solid rocket boosters weighed 83 805 kg empty each. Each of the solid rocket boosters contained 498 300 kg fuel. They burned out in 87 seconds.

Total launch weight was 2 000 000 kg. Of this weight, 1 714 605 kg was fuel, and 104 000 kg reached orbit.

As a comparison, the maximum takeoff weight of an A380 is 569 000 kg.

The Oberth Effect

The Oberth effect applies when we (our rocket) flies through a gravity field, and we have to choose where to burn our engine, at low or high altitude.

This situation applies to launches from the ground, because we have choice of burning at lower altitude, or burning at higher altitude.

Suppose that there are two places to choose from, and that between them is a difference of $d.g$.

e : Effective exhaust velocity.

m_0 mass that we start with.

m_1 mass that we end with.

v_0 initial speed.

We relate the effect of passing against gravity g over distance d.

Let m be the mass of an object. Let v_0 be the starting speed. We want to know the end speed v_1.

We start (at the low end) with kinetic energy $E = \frac{1}{2}mv_0^2$.

At the higher end, our kinetic energy equals $\frac{1}{2}mv_1^2$. The difference is dmg, the energy that was needed to pass against gravity.

So we obtain

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + dmg.$$

It follows that

$$v_1 = \sqrt{v_0^2 - 2dg}.$$

If $v_0^2 - 2dg < 0$, this means that our initial speed was not high enough to make it all the way up.

When to Burn?

First burning, then increasing the potential energy results in:
$v := v_0 + e.\log(\frac{m_0}{m_1})$.

$$v_1 := \sqrt{(v_0 + e.\log(\frac{m_0}{m_1}))^2 - 2dg}.$$

First increasing potential energy, then burning results in:
$v := \sqrt{v_0^2 - 2dg}$.

$$v_1 := \sqrt{v_0^2 - 2dg} + e.\log(\frac{m_0}{m_1}).$$

In order to compare the terms, note that they are very similar. Both contain an expresssion of form $\sqrt{X^2 - S}$. If $S = 0$, the terms are equal. Otherwise, the effect of subtracting S is smaller when X is bigger. This implies that one should first accelerate, than pass through the gravity field.

The fact that subtracting S has less effect when X is bigger can be seen from the Taylor expansion:

$$\sqrt{X^2 - S} = X\sqrt{1 - \frac{S}{X^2}} = X(1 - \frac{S}{2X^2} + \frac{S^2}{8X^4} - \frac{S^3}{16X^6}...).$$

The Oberth-effect can be intuitively explained as follows: It is a waste of energy to take fuel up, in order to burn it at high altitude, if you can burn it at a lower altitude.

Differential Equations

An equation of form $y' = F(x, y)$ is called differential equation.

A solution for the equation above is a function y of type $\mathcal{R}$ to $\mathcal{R}$, s.t. for all $x \in \mathcal{R}$,

$$y'(x) = F(y(x)).$$

For example:

$$y'(t) = c.y(t)$$

(Money on the bank, if everything goes well.)

An artificial example:

$$y'(x) = \sqrt{c.y(x)}.$$

Systems of Differential Equations

In this course, we are mostly modelling physical processes, so we assume that $y(t)$ depends on time t.

In most cases, $y(t)$ will be vector valued, i.e. of type $\mathcal{R} \rightarrow \mathcal{R}^k$ for some $n > 1$.

So, the equation gets form

$$\overline{y}(t) = \overline{F}(\ t,\ \overline{y}(t)\ ),$$

and F is of type $\mathcal{R} \times \mathcal{R}^k \rightarrow \mathcal{R}^k$.

Order of a Differential Equation

On the previous slides, the highest derivative of y that occurred in the equations, was y'. This makes the equations first-order.

In general, the order of a differential equation is defined by the highest derivative that occurs in it. E.g. $y'' = F(y, y')$ is second order.

Example:

$$y''(x) = c.\sqrt{1 + (y'(x))^2}.$$

This is the definition of a catenary, the shape that a freely hanging chain assumes.

Note that Most Grunwaldzki is not a catenary! This is because the weight per distance does not depend on the steepness.

(MG can be characterized by $y''(x) = c$.)

A higher-order differential equation can be made first-order, by increasing its dimension:

Suppose that we have the $(n+1)$-th order equation:

$$\overline{y}^{(n+1)}(t) = \overline{F}(\ \overline{y}(t), \overline{y}^{(1)}(t), \ldots, \overline{y}^{(n)}(t)\ ).$$

Define $\overline{w}_0(t) = \overline{y}(t), \ \ \overline{w}_1(t) = \overline{y}^{(1)}(t), \ldots, \overline{w}_n(t) = \overline{y}^{(n)}(t)$. Then the equation can be replaced by the system

$$\begin{cases} \overline{w}_0'(t) & = & \overline{w}_1(t) \\ \overline{w}_1'(t) & = & \overline{w}_2(t) \\ \cdots & = & \cdots \\ \overline{w}_{n-1}'(t) & = & \overline{w}_{n-1}(t) \\ \overline{w}_n'(t) & = & \overline{F}(\ \overline{w}_0(t), \overline{w}_1(t), \ldots, \overline{w}_n(t)\ ) \end{cases}$$

If the original equation had dimension k, then the new equation has dimension $k.(n+1)$.

Autonous vs. Non-Autonomous

A differential equation of form $\overline{y}(t) = \overline{F}(\overline{y}(t))$ is called autonomous. If it has form $\overline{y}(t) = \overline{F}(\ t,\ \overline{y}(t)\ )$, then it is non-autonomous.

A non-autonomous equation $\overline{y}(t) = \overline{F}(\ t,\ \overline{y}(t)\ )$ can be made autonomous by adding an extra parameter x as follows:

$$\begin{cases} x' & = & 1 \\ \overline{y}'(t) & = & \overline{F}(\ x,\ \overline{y}(t)\ ) \end{cases}$$

If the old equation has dimension k, then the new equation has dimension $k+1$.

In general, differential equations can be very hard to understand. Typically, one tries to answer the following questions:

- Is there a closed form for $y(t)$?

  (In general, a differential equation can have infinitely many solutions. Some of the solutions may be closed, while others are not.)

- Is the differential equation invariant under certain operations? (For example, rotation, mirroring, reflecting, scaling).

- Do the solutions have certain invariants?

  An invariant is a function $\overline{f}$, s.t. for all solutions $\overline{y}$, for all $t_1, t_2 \in \mathcal{R}$,

  $$\overline{f}(\overline{y}(t_1)) = \overline{f}(\overline{y}(t_2)).$$

Some More Orbit Calculations

Consider the differential equation:

$$\overline{a} = -\frac{\overline{x}}{\|\overline{x}\|^3}.$$

It is autonomous, vector valued, and second-order. In order to make it first-order, it can be replaced by:

$$\begin{cases} \overline{x}' & = & \overline{v} \\ \overline{v}' & = & -\dfrac{\overline{x}}{\|\overline{x}\|^3} \end{cases}$$

The vector notation can be replaced by:

$$\begin{cases} x_1' &=& v_1 \\ x_2' &=& v_2 \\ v_1' &=& -\dfrac{x_1}{\sqrt{(x_1^2 + x_2^2)^3}} \\ v_2' &=& -\dfrac{x_2}{\sqrt{(x_1^2 + x_2^2)^3}} \end{cases}$$

Remember that $x_1(t), x_2(t), v_1(t), v_2(t)$ are functions of time.

It can be shown that $\overline{x} \times \overline{v}$ is an invariant of the equation. It represents Kepler's area law.

It can also be shown that $\frac{1}{2}\|\overline{v}\|^2 - \frac{1}{\|\overline{x}\|}$ is an invariant. It represents preservation of energy.

Assuming that $\omega = \sqrt{\dfrac{1}{r^3}}$, it can be shown that

$$\begin{cases} x_1 & = & r.\cos\omega t \\ x_2 & = & r.\sin\omega t \\ v_1 & = & -r.\omega.\sin\omega t \\ v_2 & = & r.\omega.\cos\omega t \end{cases}$$

is a solution of the differential equation.

Let A be a matrix that preserves length. (This means that always $\|A\overline{x}\| = \|\overline{x}\|$.)

If $(\overline{x}, \overline{v})$ is a solution, then $(A\overline{x}, A\overline{v})$ is a solution.

Operations that preserve length are mirorrings and rotations, and their combinations.