Quaternions

Vector Products

Definition: Let $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$ be vectors.

The dot product (also called scalar product or inner product) of $\overline{x}$ and $\overline{y}$ is defined as

$$\overline{x} \cdot \overline{y} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

It is a real number.

The dot product can be interpreted as

$$|\overline{x}|.|\overline{y}|.\cos\varphi,$$

where φ is the angle between the vectors. (Note that $\overline{y}.\cos\varphi$ is the length of the projection of $\overline{y}$ onto $\overline{x}$.) (Show that both $\cdot$ and its interpretation are linear in both of their arguments, and that the interpretation makes sense for parallel and orthogonal vectors.)

Vector Products

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The cross product is defined as

$$\begin{pmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_1 - y_3 x_1 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$

It is written as $\overline{x} \times \overline{y}$.

Intuitively, it denotes a vector whose length equals the surface area of the parallelogram that is formed by the vectors $|x|.|y|.\sin\varphi$.

Its direction is orthogonal to both vectors, and determined by the screw driver rule.

(Show that for unit vectors, the formal definition coincides with the intuition, and that the operation is linear in both arguments.)

Quaternions

Definition A quaternion is a quadruple $(r;\ x_1, x_2, x_3)$, where $r, x_1, x_2, x_3 \in \mathcal{R}$.

The quaternion can be viewed as a quadruple of real numbers. (This is how I defined it.) In that case, the components are called 1, i, j, and k.

It can be also viewed as a pair, consisting of a real number and a vector. In that case, we call r the real or scalar part and (x_1, x_2, x_3) the vector part.

Quaternions

For $r \in \mathcal{R}$, we identify r and $(r;\ 0, 0, 0)$.

For $\overline{v} \in \mathcal{R}^3$, we identify $\overline{v}$ and $(0; \overline{v})$.

Definition: Addition, subtraction and multiplication by a real number, are defined member wise.

Multiplication

Multiplication is defined from

$$i^2 = j^2 = k^2 = ijk = -1.$$

(Hamilton wrote these equations into Brougham Bridge in Dublin on 16.10.1843.)

It can be easily checked that the following matrix follows from the equation above:

$\cdot$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Multiplication Using Dot and Cross Product

Using dot product and cross product, the product of $(r_1;\ \overline{x}_1)$ and $(r_2;\ \overline{x}_2)$ can be written as

$$(r_1 r_2 - \overline{x}_1 \cdot \overline{x}_2;\ r_1\overline{x}_2 + r_2\overline{x}_1 + \overline{x}_1 \times \overline{x}_2).$$

Quaternion multiplication is associative:

$$q_1.(q_2.q_3) = (q_1.q_2).q_3.$$

It is also distributive:

$$q_1.(q_2 + q_3) = q_1.q_2 + q_1.q_3.$$

It is not commutative.

Norm of a Quaternion

Let $q = (r;\ x_1, x_2, x_3)$ be a quaternion. The norm of q, written as $\|q\|$, is defined as $\sqrt{r^2 + x_1^2 + x_2^2 + x_3^2}$.

It is easily checked that, for any two quaternions q_1 and q_2, one has $\|q_1.q_2\| = \|q_1\|.\|q_2\|$.

Conjugate of a Quaternion

Definition: For a quaternion $q = (r;\ x_1, x_2, x_3)$, define the conjugate $\overline{q}$ as $(r;\ -x_1, -x_2, -x_3)$.

It can be checked that $q.\overline{q} = \|q\|^2$ and that $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$.

Using the first property, one can define $q^{-1} = \dfrac{\overline{q}}{\|q\|^2}$.

From the second property follows that $(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$.

Quaternions and Rotations

We are interested in quaternions because they are the most natural way to represent rotations in three dimensional space.

A rotation can also be represented by a matrix, but:

1. A quaternion is a bit more compact, and multiplying quaternions is a bit cheaper than multiplying matrices. (But the difference is not significant.)
2. It is easy to read of the rotation axis and the angle from a quaternion. Doing this for a matrix is harder.
3. A quaternion is always a well-formed rotation. A matrix may get polluted by floating point rounding, and may need correction.
4. In all cases, the simplest way to construct a rotation matrix is through the quaternion, so there is no way around them.

Quaternions and Rotations

Definition: Let $\overline{x}$ be a vector, and let φ be a real number. We define

$$q_{\overline{x},\varphi} = (\|\overline{x}\| \cos \frac{1}{2}\varphi; \quad \overline{x} \sin \frac{1}{2}\varphi).$$

Definition: For a quaternion q, the function f_q is defined as follows:

$$\lambda\overline{v} : f_q(\overline{v}) = q.\overline{v}.q^{-1}.$$

Theorem: The function $f_{q_{\overline{x},\varphi}}$ defines a rotation around axis $\overline{x}$ over angle φ.

In order to determine the direction of rotation, use the screwdriver rule or corkscrew rule. (lefty-loosey, righty-tighty)

We prove this important theorem on the next slides.

- First observe that f_q does not depend on $\|q\|$, as long as it is not zero.

- It can be easily checked that f_q is always a linear function. This means that $f_q(\lambda\overline{v}) = \lambda f_q(\overline{v})$ and $f_q(\overline{v}+\overline{w}) = f_q(\overline{v}) + f_q(\overline{w})$. As a consequence, f_q can be represented by a matrix.

- For two quaternions q_1 and q_2 and a vector $\overline{v}$, we have $f_{q_1 q_2}(\overline{v}) = f_{q_1}(f_{q_2}(\overline{v}))$. This implies that the functions can be composed by multiplying the quaternions.

- If one writes $q = (r;\overline{x})$, then $f_q = q.\overline{v}.q^{-1}$ has form

$$\frac{(r;\overline{x})(0;\overline{v})(r;-\overline{x})}{r^2+\|\overline{x}\|^2} = \frac{(-\overline{x}\cdot\overline{v};\ r\overline{v}+\overline{x}\times\overline{v})(r;-\overline{x})}{r^2+\|\overline{x}\|^2} =$$

$$\frac{(0;\ r^2\overline{v}+2r(\overline{x}\times\overline{v})+(\overline{x}\cdot\overline{v})\overline{x}-(\overline{x}\times\overline{v})\times\overline{x}\ )}{r^2+\|\overline{x}\|^2}. \tag{1}$$

Defining Rotations

We first give a direct expression for rotations. After that, we show that it is equal to the expression on the previous slide.

Assume that we want to rotate with angle φ around axis $\overline{e}$. We assume that $\overline{e}$ is a unit vector. Let $\overline{v}$ be the factor that we want to rotate: Define the following vectors:

1. Projection of $\overline{v}$ onto $\overline{e}$: $\quad \overline{V}_z = \overline{e}(\overline{e} \cdot \overline{v})$.
2. Direction in which rotation would start moving, if it would be carried out gradually: $\quad \overline{V}_y = \overline{e} \times \overline{v}$.
3. The arm of the rotation, when it starts: $\quad \overline{V}_x = (\overline{e} \times \overline{v}) \times \overline{e}$.

We have $\overline{v} = \overline{V}_x + \overline{V}_z$.

Rotation of $\overline{v}$ over angle φ results in $\overline{V}_z + \overline{V}_x \cos\varphi + \overline{V}_y \sin\varphi =$

$$(\overline{e}.\overline{v})\overline{e} + ((\overline{e} \times \overline{v}) \times \overline{e}) \cos\varphi + (\overline{e} \times \overline{v}) \sin\varphi. \qquad (2)$$

Comparing the Expressions

We replace $r := \cos \frac{1}{2}\varphi$, and $\overline{x} := \overline{e} \sin \frac{1}{2}\varphi$ in (1). The result is

$$\frac{\overline{v}\cos^2 \frac{1}{2}\varphi + 2(\overline{e} \times \overline{v}) \cos \frac{1}{2}\varphi \sin \frac{1}{2}\varphi + (\overline{e} \cdot \overline{v})\overline{e} \sin^2 \frac{1}{2}\varphi - ((\overline{e} \times \overline{v}) \times \overline{e}) \sin^2 \frac{1}{2}\varphi}{\cos^2 \frac{1}{2}\varphi + \|\overline{e}\|^2 \sin^2 \frac{1}{2}\varphi}.$$

Note that $\|\overline{e}\| = 1$, so that the denominator equals 1, and $\sin \varphi = 2. \sin \frac{1}{2}\varphi \cos \frac{1}{2}\varphi$. We get:

$$\overline{v}\cos^2 \frac{1}{2}\varphi + (\overline{e} \times \overline{v}) \sin \varphi + (\overline{e} \cdot \overline{v})\overline{e} \sin^2 \frac{1}{2}\varphi - ((\overline{e} \times \overline{v}) \times \overline{e}) \sin^2 \frac{1}{2}\varphi.$$

Using $\overline{v}\cos^2 \frac{1}{2}\phi = (\overline{V}_x + \overline{V}_z) \cos^2 \frac{1}{2}\phi$ and $\cos \varphi = \cos^2 \frac{1}{2}\varphi - \sin^2 \frac{1}{2}\varphi$, we obtain the same as (2):

$$(\overline{e} \cdot \overline{v})\overline{e} + (\overline{e} \times \overline{v}) \sin \varphi + ((\overline{e} \times \overline{v}) \times \overline{e}) \cos \varphi.$$

Matrix Representation

Since f_q is always a linear function, it is possible to give a matrix representation. Here it is, assuming that $q = (r;\ \overline{x})$:

$$\frac{\begin{pmatrix} r^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - rx_3) & 2(x_1x_3 + rx_2) \\ 2(x_1x_2 + rx_3) & r^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - rx_1) \\ 2(x_1x_3 - rx_2) & 2(x_2x_3 + rx_1) & r^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}}{r^2 + \|\overline{x}\|^2}$$

Remarks

Although the correctness proof was not so easy, quaternions are easy to use. Use them!

Coordinate Systems

Coordinate systems are always right handed, which means that $Z = X \times Y$.

In order to define a new coordinate system $\mathcal{C}'$ in terms of an existing coordinate system $\mathcal{C}$, one needs to define its origin $\overline{b}$ and orientation q.

The following transformation transforms $\mathcal{C}'$-coordinates to $\mathcal{C}$ coordinates:

$$T(\overline{x}) = \overline{b} + f_q(\overline{x}).$$

In order to transform $\mathcal{C}$-coordinates to $\mathcal{C}'$-coordinates, use

$$T^{-1}(\overline{x}) = -f_{q^{-1}}(\overline{b}) + f_{q^{-1}}(\overline{x}).$$

Coordinate Systems

Earth Centered Earth Fixed (ECEF) coordinates are defined as follows:

Origine is the center of mass of the earth.

X: From the center of the earth towards the point where the equator intersects with the 0 meridian.

Y: From the center of the earth towards the point where the equator intersects with the 90 deg meridian.

Z: From the center of the earth towards the north pole.

The institute is on position N 51 deg 6 min 39.9 sec and E 17 deg 3 min 13.4 sec . The position in ECEF is

$$(3835996.227,\ 1176715.805,\ 4941310.474).$$

Local East North Up (LENU) is defined as follows:

Origin is the point where you stand, at sea level.

X: East.

Y: North.

Z: Up.

In order to transform LENU to ECEF, relative to the institute, use $\bar{b}$ from the previous slide, and
$q = (0.560541377;\ 0.197886954,\ 0.267691243,\ 0.758271400)$.

Eye or camera coordinates are defined as follows:

The origin is the position of the camera.

X: To the right, relative to the camera's orientation.

Y: Up, relative to the camera's orientation.

Z: Behind the camera.

In order to make the perspective computation, first transform into eye coordinates using T^{-1}. After that, use:

$$(x', y') = \begin{cases} (-\frac{x}{z}, -\frac{y}{z}), & \text{if } z \leq -1 \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Of course, some additional scaling and clipping may be necessary.

In computer graphics, all transformations are represented by homogeneous or projective transformations. The result is the same.

Airplane coordinates are defined as follows:

X: Pointing forward along the frame, in flying direction, when the plane flies straight.

Y: Pointing to the right. (Starboard side.)

Z: Pointing downward.

The origin could be set in the center of mass of the plane, but this is not practical. The position of the center of mass depends on load and on fuel, and is likely to change during flight.

In addition to the previous, there may be more coordinate systems: For example for nose wheel steering, or for movable aerodynamic surfaces.

The view of the pilot is determined by

$$\overline{b} = (14, 0, -1), \quad q = (1; -1, -1, 1).$$

The view of a passenger in seat 27A (left looking, somewhat in the back of the plane) is determined by

$$\overline{b} = (-10, -2, -1), \quad q = (1; -1, 0, 0).$$

If an airplane flies west at an altitude of 5000 meter over the origin of a LENU coordinate system, then the coordinate system of the plane is determined by

$$\overline{b} = (0, 0, 5000), \quad q = (0; 0, 1, 0).$$

How to Obtain Coordinate Transformations

It is sometimes difficult to understand what the transformation

$$T(\overline{x}) = \overline{b} + f_q(\overline{x})$$

means.

It has two meanings:

1. The position of a rigid object or a camera.
2. A change of coordinate system from internal coordinates of the object or camera to outside coordinates.

Assume that the rigid object or camera has an internal coordinate system $\mathcal{C}_2$. We want to express its position in some other coordinate system $\mathcal{C}_1$.

Start by positioning the object in such a way that its origin equals the origin of $\mathcal{C}_1$, and the XYZ-axes of $\mathcal{C}_2$ are aligned with $\mathcal{C}_1$.

In this position, we have we have $T(\overline{x}) = (0,0,0) + f_{(1;0,0,0)}(\overline{x})$ and the two coordinate systems are the same.

Now we first move the object to the position where we want it to stand, and then rotate it around its origin (after movement) into its proper orientation.

The pair $(\overline{b}, q)$ represents the position of the object. At the same time, the function $T(\overline{x}) = \overline{b} + f_q(\overline{x})$ is a function that transforms coordinates: If $\overline{x}$ is the position of a point expressed in coordinate system $\mathcal{C}_2$, then $T(\overline{x})$ is the position of the same point expressed in coordinate system $\mathcal{C}_1$.

Composition of Movements

A lot of confusion appears if one wants to make two or more movements with the object. We first move over $(\overline{b}_1, q_1)$, and after that over $(\overline{b}_2, q_2)$.

1. If the second movement is still expressed in the original coordinate system $\mathcal{C}_1$, one can easily build the transformation $T(\overline{x}) = (\overline{b}_1 + \overline{b}_2) + f_{q_2.q_1}(\overline{x})$.

2. If the second movement is expressed in the internal coordinate system of the object after the first move, one obtains: $T(\overline{x}) = T_1(T_2(\overline{x})) = \overline{b}_1 + f_{q_1}(\overline{b}_2) + f_{q_1.q_2}(\overline{x})$.

   The correctness can be seen from the second meaning of $T(\overline{x})$. Let $\mathcal{C}_2$ be the coordinate system of the object after its first move. Let $\mathcal{C}_3$ be the position after the second move. Then T_2 transforms $\mathcal{C}_3$ to $\mathcal{C}_2$, and T_1 transforms from $\mathcal{C}_2$ to $\mathcal{C}_1$. It follows that $T_1 \circ T_2$ transforms $\mathcal{C}_3$ to $\mathcal{C}_1$.

Consider a passenger sitting in seat 27A, looking out of the window: For the passenger in 27A, the matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

It often happens that one knows the orientation q, and one knows that $T(\overline{x}_0) = \overline{y}_0$.

In that case, $\overline{b}$ can be solved from $\overline{y}_0 = \overline{b} + f_q(\overline{x}_0)$.

The result $\overline{b} = \overline{y}_0 - f_q(\overline{x}_0)$.

This happens when one knows the position of the center of mass of the plane, and its orientation.

Angular Speed

Angular velocities are represented by a vector $\overline{\omega}$ that is aligned along the axis of rotation, whose direction is determined by the cork screw rule, and whose length is determined by the rotation speed. One rotation per second means that $\overline{\omega}$ has length 2π.

If the rotation is around the origin, then the speed of a point $\overline{x}$ can be expressed by

$$\overline{v}(\overline{x}) = \overline{\omega} \times \overline{x}.$$

If the rotation is not around the origin, then let $\overline{b}$ be the center of rotation. The speed of point $\overline{x}$ can be expressed by

$$\overline{v}(\overline{x}) = \overline{\omega} \times (\overline{x} - \overline{b}) = \overline{b} \times \overline{\omega} + \overline{\omega} \times \overline{x}.$$

Speed of Rigid Objects

A rigid object is an object, whose components always keep the same orientation to each other.

The speed function $\overline{v}(\overline{x})$ has form

$$\overline{v}(\overline{x}) = \overline{w} + \overline{\omega} \times \overline{x}.$$

If we know ω and that a certain point $\overline{x}_0$ has speed $\overline{w}_0$, then we find

$$\overline{w} = \overline{w}_0 - \overline{\omega} \times \overline{x}_0.$$

When $\overline{x}_0$ is far from the origin, it may be better to keep the representation

$$\overline{v}(\overline{x}) = \overline{w}_0 + \overline{\omega} \times (\overline{x} - \overline{x}_0),$$

because both $\overline{w}$ and $\overline{x}$ will become big and waste floating point accuracy. This may happen when you want to fly around the world or to the moon. I will ignore this problem.

Effect of Rigid Speed Function on Position and Orientation

Let $\mathcal{C}$ be a coordinate system, defined by origin $\overline{b}$, and orientation q.

We want to know how $\mathcal{C}$ changes when it has a speed, and a rotation.

We assume speed function $\overline{v}(\overline{x})$, as defined on the previous slide.

The origin has speed $\overline{v}(\overline{b})$. After small time h, it will be at position

$$\overline{b}' = \overline{b} + h.\overline{v}(\overline{b}).$$

In small time h, angular speed $\overline{\omega}$ causes rotation over angle $\|\overline{\omega}\|h$. The quaternion of this rotation is $(\ \|\overline{\omega}\| \cos \frac{1}{2}h\|\overline{\omega}\|;\ \ \overline{\omega} \sin \frac{1}{2}h\|\overline{\omega}\|\ )$. This implies that

$$q' = (\ \|\overline{\omega}\| \cos \frac{1}{2}h\|\overline{\omega}\|;\ \ \overline{\omega} \sin \frac{1}{2}h\|\overline{\omega}\|\ )\ q.$$

Expressing a Rigid Speed Function in Different Coordinates

Let coordinate system $\mathcal{C}'$ be defined in $\mathcal{C}$ by origin $\bar{b}$ and orientation q. Let T be the transformation function:

$$T(\overline{x}) = \bar{b} + f_q(\overline{x}).$$

Assume that we have a speed function $\overline{v}'(\overline{x}) = \overline{w}' + \overline{\omega}' \times \overline{x}$, which is defined in coordinate system $\mathcal{C}'$. We want the corresponding speed function $\overline{v}$ expressed in $\mathcal{C}$.

Define

$$\overline{\omega} = f_q(\overline{\omega}').$$

We have $\overline{v}'(\overline{0}) = \overline{w}'$, so that $\overline{v}(T(\overline{0})) = f_q(\overline{w}')$.

It follows that

$$\overline{v}(\overline{x}) = (\ f_q(\overline{w}') - \overline{\omega} \times \bar{b}\ ) + \overline{\omega} \times \overline{x}.$$

Determining Wind Speed

Suppose that a rigid object (an airplane) has position $T(\overline{x}) = \overline{b} + f_q(\overline{x})$, and speed $\overline{v}(\overline{x}) = \overline{w} + \overline{\omega} \times \overline{x}$.

We want to express speed in internal coordinates, using $\overline{v}'(\overline{x}) = \overline{w}' + \overline{\omega}' \times \overline{x}$. We start with:

$$\overline{\omega}' = f_{q^{-1}}(\overline{\omega}).$$

We also know that $\overline{v}(\overline{b}) = \overline{w} + \overline{\omega} \times \overline{b}$. It follows from $\overline{v}'(\overline{0}) = \overline{w}'$ that

$$\overline{w}' = f_{q^{-1}}(\overline{w}) + f_{q^{-1}}(\overline{\omega} \times \overline{b}) = f_{q^{-1}}(\overline{w}) + \overline{\omega}' \times f_{q^{-1}}(\overline{b}).$$

This together gives the complete definition of $\overline{v}'(\overline{x}) = \overline{b}' + \overline{\omega}' \times \overline{x}$.

The wind that you feel when standing on point $\overline{x}$ (still in internal coordinates), equals

$$-\overline{v}'(\overline{x}).$$

The previous expression is very important. You need it all the time for computing aerodynamic forces, (wings, etc.) and frictional forces. (wheels, etc.)