# Three Dimensional Fligh
# Model

## Body Coordinates

Aircraft Body (ABC) coordinates are fixed to the aircraft
They move and rotate with the aircraft.

The center is in the center of mass of the airplane. The X
points forward. The Y-axis points to the right. The Z-axi
downward.

# Symmetry in the Inertial Matrix

The inertial matrix of a point mass $(m, \overline{x})$ equals $m\,\overline{x} \times (\overline{\omega}$

Writing $\overline{x} = (x_1, x_2, x_3)$, results in

$$m \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{pmatrix}$$

For the total airplane, one would get:

$$I = \int_{\overline{x} \in \mathcal{R}^3} m(\overline{x}) \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{pmatrix}$$

where $m$ is the mass density function.

## Symmetry

Nearly all airplanes are left-right symmetric. This means
$m(x_1, x_2, x_3) = m(x_1, -x_2, x_3)$. It follows that all entries v
single, non-squared occurrence of $x_2$ are equal to 0.

$$\begin{pmatrix} I_{1,1} & 0 & I_{1,3} \\ 0 & I_{2,2} & 0 \\ I_{1,3} & 0 & I_{3,3} \end{pmatrix}.$$

In addition, $I_{1,3}$ is usually much smaller than the other te
that it is often reasonable to neglect it. This is an (estima
me) inertial matrix for the B737:

$$\begin{pmatrix} 1070647 & 0 & 263278 \\ 0 & 2027731 & 0 \\ 263278 & 0 & 2840144 \end{pmatrix}$$

## Environment Coordinate Systems

There are many ways to fix a coordinate system for the environment. The two main ones are:

1. Earth Centered Earth Fixed (ECEF) coordinates. Th
   is in the mass center of the earth. The X-axis points t
   the equator and the 0-meridian. The Y-axis points th
   equation and the 90-meridian. The Z-axis points thro
   North Pole.

2. Local East North Up (LENU) coordinates. The cente
   the surface of the earth, close to some point of interes
   X-axis points east from the point of interest. The Y-a
   north from the point of interest. The Z-axis points up
   LENU coordinates usually assume that the earth is fl

## Representing Aircraft Position

Aircraft position can be represented by a position vector $\overline{x}$
center) and a quaternion $q$.

The function $\lambda\overline{v}\ f_q(\overline{v}) + \overline{x}$ transforms internal coordinates
to external coordinates, either LENU or ECEF.

In the rest of the slides, we assume LENU.

The transformation $\lambda\overline{v}\ f_q(\overline{v}) + \overline{x}$ is often called rototrans.
(Rotation + Translation).

## Attitude Angles

Attitude angles are usually denoted as follows: Direction:
$\theta$. Roll: $\psi$.

- First rotate the plane (in the horizontal plane) in the
flying direction. This gives angle $\phi$. We define turning
positive.
- After that, pitch the plane. This gives angle $\vartheta$. Pitchi
positive.
- Finally, roll the plane. This results in angle $\psi$. Rolling
wing down is positive.

# Representing Attitude Angles as Quaternion

Remember that $q_{\overline{x},\alpha}$ denotes the quaternion that represen
rotation around axis $\overline{x}$ over angle $\alpha$. Also remember that
quaternion multiplication $q_1.q_2$ represents composition of
(first $q_2$, then $q_1$.)

Then

$$q(\phi, \vartheta, \psi) = \; q_{\overline{Z},\phi}.\, q_{\overline{Y},\vartheta}.\, q_{\overline{X},\psi}$$

The vectors $\overline{X}, \overline{Y}, \overline{Z}$ are the unit axis vectors, i.e. $\overline{X} = (1$

$q(\phi, \vartheta, \psi)$ is the quaternion that the rotations $\phi, \vartheta, \psi$ in pr
aircraft body coordinates.

In order to get a transformation to LENU, use $(0; 1, 1, 0).$

## Representing Attitude Angles as Matrix

It is easy to represent $q(\phi, \vartheta, \psi)$ as matrix, because all rot
around coordinate axes:

$$q_{\overline{Z},\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Representing Attitude Angles as Matrix (2)

$$q_{\overline{Y},\vartheta} = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}$$

## Representing Attitude Angles as Matrix (3)

$$q_{\overline{X},\psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

The complete rotation matrix equals:

$$\begin{pmatrix} \cos\phi\cos\vartheta & -\sin\phi\cos\psi + \cos\phi\sin\vartheta\sin\psi & \sin\phi\sin\psi + \cos\phi\sin\vartheta\cos\psi \\ \sin\phi\cos\vartheta & \cos\phi\cos\psi + \sin\phi\sin\vartheta\sin\psi & -\cos\phi\sin\psi + \sin\phi\sin\vartheta\cos\psi \\ -\sin\vartheta & \cos\vartheta\sin\psi & \cos\vartheta\cos\psi \end{pmatrix}$$

## Recovering the Attitude Angles

The matrix on the previous slide can be used to recover th
angles. In order to recover the angles, write the matrix in

$$q(\phi, \vartheta, \psi) = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix}.$$

$$\begin{cases} \phi = \arctan(A_{2,1}, A_{1,1}) \\ \vartheta = \arctan(-A_{3,1}, \sqrt{A_{1,1}^2 + A_{2,1}^2}) \\ \psi = \arctan(A_{3,2}, A_{3,3}) \end{cases}$$

Recovering the angles is useful if you want to analyse sim
manouvers, like turns.

## Representing Attitude Angles as Matrix (4)

The remaining quaternion $(0; 1, 1, 0)$ has rotation matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

## State of the Airplane

At a given time point in time, the state of the airplane is
determined by a position vector $\overline{x}$, a speed vector $\overline{v}$, an or
quaternion $q$, and an angular velocity vector $\overline{\omega}$.

In addition, the various components of the airplane may h
parameters, which may be important for stability analysis

If the plane has an autopilot, it may have some internal st
well, which is important for stability analysis.

We need expressions for $\overline{F}$ and $\overline{T}$. Using those, one can ge
differential equation for the complete airplane.

We assume that $\overline{F}$ and $\overline{T}$ are in body coordinates.

## Aircraft model in internal Coordinates

Gravity in aircraft coordinates:

$$\overline{F}_G = f_{q^{-1}}(0, 0, -M.g).$$

Gravity does not create torque around center of mass.

## Engines

It is assumed that the engines produce a force in forward
at least that's what they are supposed to do.

If one want to be able to model asymmetric flight, the eng
to be treated separately. On the B737, the engines have p
$\overline{e}_1 = (2, -5, 2)$, $\overline{e}_2 = (2, 5, 2)$. We have

$$\begin{cases} \overline{F} = (1,0,0)(F_{thr,1} + F_{thr,2}) \\ \overline{T} = \overline{e}_1 \times (1,0,0)F_{thr,1} + \overline{e}_2 \times (1,0,0)F_{thr,2} \end{cases}$$

As with 2D, I have no real clue how to get reasonable valu
$F_{thr,1}$ and $F_{thr,2}$.

## Main Wings

I have no good idea how wings should be modeled in 3D.

First observe that, when the plane has angular velocity, d parts of the wing will have different speed and angle of att must be modeled properly, because fundamental aspects o behavior of the plane depend on it. (The tendency to stop to increase it if parts of the wing are stalled.)

This problem can be solved by cutting the wing in slices, modeling the slices separately.

This immediately causes a fundamental problem: Do the slices influence each other? Probably yes, but since we ha how, we ignore this problem.

In the rest, we speak only about wing slices, which we ass independent.

Next major problem is how to deal with sweep back and s
Perhaps sweep back can be handled as a special way of si

Sideslip makes the chord length longer and decreases aspe
It does not change the acting point (if we assume that wi
don't influence each other).

We assume that each wing segment has a position $\overline{x}_s$, a s
angle $\phi_s$, a pitch angle $\vartheta_s$ and a dihedral angle $\psi_s$. Since
usually have positive dihedral, segments on the right wing
have negative $\psi_s$, while those on the left wing probably h
positive $\psi_s$.

Purpose of sweep back is to decrease Mach number. This
that air flow is (at least partially) insensitive to side slip.
to take this into account.

A second problem is that it is not clear which direction li
have. It depends on which direction the air flow takes ove
wing, which we don't know.

It is closely related to the question, whether different win
influence each other.

Ignoring all these big unknowns, we proceed:

We assume that the wing is separated into a couple of seg
(20?), and treat each of the segments separately:

Each slice has a separate position, separate $\phi_s, \vartheta_s$, and di

The rotation from airplane body coordinates to coordinat
sequent equals $q^{-1}(\phi_w, \vartheta_w, \psi_w)$. This means that
$\overline{v}' = f_{q^{-1}}(\phi_w, \vartheta_w, \psi_w)(f_{q^{-1}}(\overline{v}))$ is the air speed in wing se
coordinates.

At this point, we cannot longer postpone a decision about
We assume that air flow does not notice side slip at all, w
implies that we can ignore the $v'_2$ coordinate.

We can now determine $\alpha$ :

$$\alpha = \arctan(-\overline{v}'_3, -\overline{v}'_1).$$

We use

$$V = \sqrt{{v'}_1^2 + {v'}_3^2}.$$

The lift- and drag- and moment coefficients equal
$C_L(\alpha), C_D(\alpha), C_M(\alpha)$. In order to determine direction of

$$\overline{L} = \frac{\overline{v}' \times (\overline{v}' \times (0,0,1))}{\| \, \overline{v}' \times (\overline{v}' \times (0,0,1)) \, \|}.$$

For the direction of drag, use

$$\overline{D} = \frac{(0,0,1) \times (\overline{v}' \times (0,0,1))}{\| \, (0,0,1) \times (\overline{v}' \times (0,0,1)) \, \|}.$$

This is probably an oversimplification, because one has to
distinguish between form drag and skin drag. The skin dr
be dependent side slip, while form drag wouldn't be.

Anyway, we write:

$$\overline{F}_w = \frac{1}{2}\rho V^2 S(\ C_L(\alpha).\overline{L} + C_D(\alpha).\overline{D}\ )$$

In order to determine torque of the segment, we need to k
position $\overline{s}$. For simplicity, we take $\overline{s}$ as the point that was
defining $C_M(\alpha)$. Then torque around $\overline{w}$ equals

$$M_w = \frac{1}{2}\,\rho\, S\, c\, V^2 C_M(\alpha).$$

Torque around the mass center can be obtained from

$$T_w = M_w + \overline{s} \times \overline{F}_w.$$

(I think that we can ignore the fact that acting point mov because of parallelogram rule.)

## Elevators and Rudder

As far as I can see, elevators and rudder can be treated in
way as the main wings.

## Fuselage

Modeling fuselage is difficult: The fuselage causes drag an

Lift has a center of pressure. The pressure center is proba front of the mass center. As a consequence, the fuselage c to pitch instability. (This effect also exists in the dimensio are currently ignoring.)

Since the fuselage is much thicker than an airfoil, we need the pressure center for drag. It is safe to assume that it is middle at 0 angle of attack, but I have no idea, where it i angles of attack. Its position may be important for stabili

For wings, one can find some data in books or on the inte found nothing useful yet for the fuselage.

## Wheels

If the wheels touch the ground, they may create force, wh
may wish to model.

If they are not on the ground (and not retracted), the wh
create resistance, which has to be modelled.

Wheels are complicated. I will discuss them later.