Stability Analysis

Stable Flight

We will study the behaviour of the airplane during stable flight.

A plane in stable flight is in equilibrium. We are interested in what happens, when this equilibrium is disturbed.

The techniques that we are explain, are general. They apply to all systems that can be described by differential equations on their state.

The techniques can also be used for the design of autopilots (systems that automatically keep a given speed, altitude, or attitude.)

State Vector

We consider only the two-dimensional flight model.

In the two-dimensional model, the state S has the following form:

- Position vector $\overline{x}$.

- Speed vector $\overline{v}$.

- Orientation (pitch) angle θ.

- Pitch rate ω.

S is a $2 + 2 + 1 + 1 = 6$-dimensional vector.

In three dimensions, state would be 12-dimensional. (It depends on which dimension you attach to the quaternion.)

The airplane model (or the simulation) defines a function F that maps S to its derivative S'. The derivative S' consists of the following components:

- Linear speed vector $\overline{v}$,

- linear acceleration vector $\overline{a}$,

- rotation vector ω,

- angular acceleration α.

The differential equation has form $S' = F(S)$.

Equilibrium

A airplane in stable flight is in equilibrium. There exist three types of equilibrium:

stable: As long as the disturbance is not too big, the system will return to the equilibrium, usually after some swinging. Examples: A weight hanging on a rope, a floating ship.

unstable: Instable equilibrium. When disturbed, the system will start moving away from the equilibrium. Example: A pencil standing on its point, a submarine under the water.

Unstable systems need active control to be kept in the equilibrium.

neutral: When disturbed, the system is in a new equilibrium. Example: A ball on a flat floor.

Equilibrium State

In equilibrium, the derivatives must be zero.

If we have a state vector of form $(\overline{x}, 0, \theta, 0)$, then $F(\overline{x}, 0, \theta, 0)$, must be equal to $(0, 0, 0, 0)$.

Equilibrium?

Note that the definition on the previous page does not capture all flights that might be considered equilibrium.

For example (in 3-dimensional case), a horizontal turn is a kind of equilibrium. During a turn, $\overline{\omega}$ is a constant vector of form $(0, 0, r)$, and the Z-coordinates of $\overline{x}$ and $\overline{v}$ are constant.

Stability

Instead of using equilibrium, one can use stability of solutions:

Let F be a transition function. We call a function s a solution of F, starting at t_0, if

$$\forall t \geq t_0, \quad s'(t) = F(s(t)).$$

If for a given solution s_1 and a time point t_0, there exists a real number d, s.t. for every solution s_2 with $\|s_1(t_0) - s_2(t_0)\| \leq d$, for every d' there exists a $t \geq t_0$, s.t. for every $t' \geq t$, we have $\|s_1(t') - s_2(t')\| \leq d'$, then we call s_1 stable at time t_0.

Stability (2)

You change a little bit in the state of the system, and you tell how small you want the difference to become.

After that, if you wait long enough, the difference will be as small as you wanted, and stay so small forever after.

Famous examples of non-stable systems are the weather, and the three-body problem.

Linear Approximation

We will consider only equilibrium states, solutions $\overline{s}(t)$ that are mostly constant.

Let $\overline{s}$ be such a solution, we know that at every moment t, we have $\overline{s}'(t) = F(\overline{s}(t))$.

We are interested in solutions close to $\overline{s}(t)$, so we express them in form $\overline{s}(t) + \overline{y}(t)$, where $\overline{y}(t)$ is the difference between the equilibrium and the concrete state.

We have

$$\overline{s}'(t) + \overline{y}'(t) = F(\ \overline{s}(t) + \overline{y}(t)\ ). \tag{1}$$

We develop $F()$ in a multivariate Taylor series relative to $\overline{s}(t)$:

$$F(\ \overline{s}(t) + \overline{y}(t)\ ) = F(\ \overline{s}(t)\ ) + \sum \left\{ \begin{array}{c} \frac{\partial F}{\partial 1}(\ \overline{s}(t)\ )\ y_1(t) \\ \frac{\partial F}{\partial 2}(\ \overline{s}(t)\ )\ y_2(t) \\ \dots \\ \frac{\partial F}{\partial 6}(\ \overline{s}(t)\ )\ y_6(t) \end{array} \right\} + O(\|\overline{y}\|^2).$$

Using $\overline{s}'(t) = F(\overline{s}(t))$ in Equation 1, we obtain:

$$\overline{y}'(t) = \sum \left\{ \begin{array}{c} \frac{\partial F}{\partial 1}(\ \overline{s}(t)\ )\ y_1(t) \\ \frac{\partial F}{\partial 2}(\ \overline{s}(t)\ )\ y_2(t) \\ \dots \\ \frac{\partial F}{\partial 6}(\ \overline{s}(t)\ )\ y_6(t) \end{array} \right\} + O(\|\overline{y}\|^2).$$

In general, because we are looking at stable states. the $\frac{\partial F}{\partial i}$ are constant.

Moreover, if we manage to keep $\|\overline{y}(t)\|$ small enough, we can ignore the quadratic and further terms.

(There may be a better argument, but this is how engineers reason.)

Homogeneous, Linear Systems with Constant Coefficients

We have obtained a differential equation of form

$$\overline{y}(t) = \begin{pmatrix} \frac{\partial F_1}{\partial 1} & \cdots & \frac{\partial F_1}{\partial 6} \\ \frac{\partial F_2}{\partial 1} & \cdots & \frac{\partial F_2}{\partial 6} \\ & \cdots & \\ \frac{\partial F_6}{\partial 1} & \cdots & \frac{\partial F_6}{\partial 6} \end{pmatrix} \overline{y}(t).$$

$\frac{\partial F_i}{\partial j}$ specifies how the i-th component of the result of F reacts to changes in the j-th component of its input, near $\overline{s}(t)$.

A differential equation of this form ($\overline{y}'(t) = A\overline{y}(t)$) is called a homogeneous, linear system with constant coefficients.

(It would be non-homogeneous if it would be of form $\overline{y}'(t) = A\overline{y}(t) + \overline{B}$)

Solving such differential equations is easier than it seems.

We want to solve $\overline{s}'(t) = A(\overline{s}(t))$. We assume that solutions have form $\overline{s}(t) = \overline{c}.e^{\lambda t}$, where $\overline{c}$ is a vector, λ is a complex number.

Substitution in the equation gives

$$\lambda \overline{c}.e^{\lambda t} = A(\overline{c}.e^{\lambda t}).$$

Since $e^{\lambda t}$ is a non-zero (complex) number, we can divide it out of the equation:

$$\lambda \overline{c} = A(\overline{c}).$$

Vectors with this property are called eigenvectors of the matrix A. The corresponding (complex) numbers are called the eigenvalues of the matrix.

Eigenvalue solvers are available in Matlab, or on the internet.

Understanding Eigenvectors

As said before, complex solutions of the differential equation have form $\overline{y}(t) = \mu.\overline{e}.e^{\lambda t}$, for $\mu \in \mathcal{C}$, and $\lambda, \overline{e}$ an eigenvalue with corresponding eigenvector.

In order to obtain solutions that are real at every moment t, one has to combine pairs of conjugate eigenvectors.

Combining Conjugate Pairs

Let λ_1, λ_2 be a conjugate eigenvalue pair. Let $\overline{e}_1, \overline{e}_2$ be the corresponding eigenvectors.

Solutions of the differential equation have form
$$\overline{y}(t) = \mu_1.\overline{e}_1.e^{\lambda_1 t} + \mu_2.\overline{e}_2.e^{\lambda_2 t}, \quad \mu_1, \mu_2 \in \mathcal{C}.$$

Because λ_1 is conjugate to λ_2, and $\overline{e}_1$ is conjugate to $\overline{e}_2$, also $\overline{e}_1 e^{\lambda_1 t}$ is conjugate to $\overline{e}_2 e^{\lambda_2 t}$.

It can be easily checked that, in order to be a real-valued function, μ_1 must be conjugate to μ_2.

The complex components will cancel each other, and the resulting function will be
$$\overline{y}(t) = 2\mathbf{Re}(\ \mu_1.\overline{e}_1.e^{\lambda_1 t}\ ).$$

$\mathbf{Re}(c)$ denotes the real part of c.

Polar Coordinates

We will represent complex numbers in polar coordinates. The pair $(r, \phi) \in \mathcal{R} \times \mathcal{R}$ represents the complex number $r. \cos \phi + i.r \sin \phi$.

Multiplication of complex numbers in polar coordinates is easy: $(r_1, \phi_1).(r_2, \phi_2) = (r_1.r_2, \ \phi_1 + \phi_2)$.

If λ_1 has form $a + bi$ (not in polar coordinates), then $e^{a+bi} = (e^a, b)$ (in polar coordinates).

If $\overline{e}_1 = (\ (r_1, \phi_1), \ldots, (r_n, \phi_n)\ )$ and $\mu = (s, \psi)$ in polar coordinates, then

$$\overline{y}(t) = 2\, e^{a\,t}\, s\, \mathbf{Re}(\ (r_1, \ \phi_1 + \psi + b\,t), \ldots, (r_n, \ \phi_n + \psi + b\,t)\ )$$

$$= 2\, e^{a\,t}\, s\, (r_1\, \cos(\phi_1 + \psi + b\,t), \ldots, r_n\, \cos(\phi_n + \psi + b\,t)\ ).$$

Understanding the Components

The eigenvalue $a + bi$ consists of two parts: a represents the damping. If $a < 0$, then the oscillation is damped. If $a > 0$, the oscillation diverges. If $a = 0$, the oscillation stays the same.

We usually want $a < 0$, and not too close to 0.

When expressed in polar coordinates, the eigenvector represents the oscillation of the system connected to $a + bi$.

$(r_1, \ldots, r_n)$ represent the relative amplitudes in the oscillation, and the $(\phi_1, \ldots, \phi_n)$ the relative phases.

Example

We try a simple, second order differential equation:

$$y''(t) = -3y(t) - y'(t). \tag{2}$$

It is clear from itse form that its solutions are damped oscillations. The characteristic polynomial is $x^2 + x + 3 = 0$. The zeroes of the polynomial are $\frac{-1 \pm i.\sqrt{11}}{2}$.

This means that complex solutions of the differential equations have form

$$y(t) = \mu_1 e^{s_1 t} + \mu_2 e^{s_2 t},$$

where s_1, s_2 are the solutions above.

Example (2)

In order to be real valued everywhere, μ_1 and μ_2 have to be conjugates.

The real valued solutions of Equation 2 have form:

$$y(t) = e^{-\frac{1}{2}t}\left(\ A\cos\frac{t}{2}\sqrt{11} + B\sin\frac{t}{2}\sqrt{11}\ \right).$$

Example (3)

Equation 2 is second order. In order to make it first-order, we use a vector (y_1, y_2), where y_2 is the derivative of y_1.

We obtain

$$
\begin{cases}
y_1' &= y_2 \\
y_2' &= -3y_1 - y_2
\end{cases}
$$

In matrix notation

$$
\overline{y}' = \begin{pmatrix} 0 & 1 \\ -3 & -1 \end{pmatrix} \overline{y}(t).
$$

Example (4)

`http://www.bluebit.gr/matrix-calculator/` gives the following eigenvalue/vector pairs:

$$-0.5 + 1.658312i \quad (-0.144337567 - 0.478713554i, \ 0.866025404)$$

$$-0.5 - 1.658312i \quad (-0.144337567 + 0.478713554i, \ 0.866025404)$$

The numbers $-0.5 \pm 1.658312i$ are the same as before $\frac{-1 \pm i.\sqrt{11}}{2}$.

We express the first eigenvector in polar coordinates:

$$(\ (0.5, -106.779 \, \mathrm{deg}), \quad (0.866025, 0 \, \mathrm{deg})\ ).$$

We see that $y_2(t)$ has an amplitude that is $\sqrt{3}$ times bigger than the amplitude of $y_1(t)$, and that it is 106.779 deg ahead of $y_1(t)$.

One can can verify this by taking the derivative of

$$y(t) = e^{-\frac{1}{2}t}(\ A\cos\frac{t}{2}\sqrt{11} + B\sin\frac{t}{2}\sqrt{11}\ ).$$

Application two Aircraft Model

The two-dimensional airplane model, that is used in Exercise 8, has a stable state at $\overline{v} = (105.468,\ 0.816,)m.s^{-1},\ \ \theta = 0.145\,\text{deg}\,.$

I collected the matrix A by making small changes in the speed of $1\ m.s^{-1}$ and in the angles $\theta, \omega,$ of $10^{-3}\,\text{deg}\,.$

The resulting matrix of partial derivatives is on the next slide.

$$\begin{pmatrix} \overline{v} \\ \overline{a} \\ \omega \\ \alpha \end{pmatrix} = A \begin{pmatrix} \overline{x} \\ \overline{v} \\ \theta \\ \omega \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.005941 & -0.07019 & -1.51203 & 0.042445 \\ 0 & 0 & 0.195168 & -1.09076 & 112.686 & 3.64708 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0008299 & 0.0638056 & -6.69163 & -1.45661 \end{pmatrix}$$

There is a lot of information in this matrix. We see that $\overline{x}$ is ignored. We see that $\overline{v}$ is copied into $\overline{x}'$, and ω into θ'.

`http://www.bluebit.gr/matrix-calculator/` gives the following eigenvalue/vector pairs: (Note that the vectors are given as columns.)

Eigenvalue 0 with eigenvectors $(1, 0, 0, 0, 0, 0)$ and $(0, 1, 0, 0, 0, 0)$ represent the fact that the behaviour of the airplane does not depend on its position.

Eigenvalues $-0.00886 \pm 0.1131i$ with eigenvectors

$$
\begin{pmatrix}
-0.01916 \mp 0.59494i & = & (0.595248, \ -91.8446\,\text{deg}) \\
-0.79560 & = & (0.7956, \ 180\,\text{deg}) \\
0.06746 \pm 0.00311i & = & (0.0675316, \ 2.63955\,\text{deg}) \\
0.00705 \mp 0.08999i & = & (0.0902657, \ -85.5205\,\text{deg}) \\
0.00004 \mp 0.00086i & = & (0.00086093, \ -87.337\,\text{deg}) \\
0.00010 \pm 0.00001i & = & (0.000100499, \ 5.71059\,\text{deg})
\end{pmatrix}
$$

This eigenvector corresponds to the phugoid. It seems to have half value time of $\ln(\frac{1}{2})/0.00866 = 78s$. This is much faster than observed. The period seems to be $\frac{2\pi}{0.113} = 55s$. This is quite consistent with the observation in the program.

Eigenvalues $-1.267793 \pm 2.532092i$ with eigenvectors

$$
\begin{pmatrix}
0.008594 \mp 0.004301i & = & (0.00961017, \ -26.5864\,\text{deg}) \\
-0.148693 \mp 0.296977i & = & (0.332122, \ -116.597\,\text{deg}) \\
-0.000004 \pm 0.027212i & = & (0.027212, \ 90.0084\,\text{deg}) \\
0.940485 & = & (0.940485, \ 0\,\text{deg}) \\
0.000336 \pm 0.021959i & = & (0.0219616, \ 89.1234\,\text{deg}) \\
-0.056029 \mp 0.02699i & = & (0.0621909, \ -154.279\,\text{deg})
\end{pmatrix}
$$

This eigenvector corresponds to the short term oscillation It seems to have half value time of $\ln(\frac{1}{2})/-1.2678 = -0.54s$. The period seems to be $\frac{2\pi}{2.53209} = 2.4814s$. This is probably consistent with the observation.