Two Dimensional Flight Model

Two Dimensional Modeling

We will study how to model aicraft in two dimensions.

I do this because two-dimensional modeling is technically a lot easier than three-dimensional modeling, while at the same time most of the technical and theoretical problems are already present.

As a consequence, two-dimensional modeling is a good preparation for three-dimensional modeling.

What can be obtained by two-dimensional modeling

Surprisingly much can be modeled in two dimensions:

- Performance analysis: Maximum speed, maximum altitude, maximum take off weight, maximum rate of climb, how much runway is needed for takeoff, how much is needed for landing, how far can the plane glide without engines. How much fuel does it need.
- Stability analysis: How does the plane react to small disturbances in altitude, attitude or speed? How hard is it for the pilot to maintain a certain speed and altitude?

  What happens when cargo is not properly loaded. How well does an automatic speed and/or altitude control system work? How to design one that works well?
- Even some stunts can be analyzed: Loopings and zero-gravity flights.

Airfoils/Wings

Before we can do any modeling, we need to study some of the basic properties of airfoils.

All passive objects have resistance in an air flow. This fact is rather intuitive.

In addition most objects also have a force that is perpendicular to the air flow. This force is called lift.

Forms that are designed to optimize the ratio Lift/Drag are called air foils.

Distinction Lift/Drag

The decomposition into lift and drag is not an arbitrary decomposition into independent axes.

In potential flow theory, lift and drag have fundamentally different causes.

In this theory, drag is caused by removal of air from the stream. (being a sink).

Lift is caused by circulation that is present in the stream.

The circulation theory is able to explain lift pretty well.

Unfortunately, potential flow theory does not work well for drag, because it predicts that passive objects have no drag. This is obviously different from observation.

Simple Observations

We see from the graph that lift depends on angle of attack.

Lift increases (linearly) with angle of attack until a maximum value is reached. After that, the lift decreases.

The decreasement of lift at high angles of attack is called stalling. It is a complicated phenomenon.

It is (somewhat) non-deterministic, and it has a memory.

A wing may be stalled at some places, and laminar at other places.

Aspect Ratio

The span of a wing is its sidewards length. The chord length is its length in flying direction.

The aspect ratio of a wing is defined as span squared divided by surface area. (span divided by average chord length.)

As a general rule, planes that are designed to fly slow have high aspect ratios, while planes that are designed to fly fast have low aspect ratios.

A wing with low aspect ratio is technically easier to construct, but wings with low aspect ratio loose lift at the tips. This effect is smaller when the plane flies faster.

Basic Formulas

Let S be the surface area of the wing segment that we are considering.

Dynamic pressure q is defined as $q = \frac{1}{2}\rho V^2$. Here ρ is the air density, (which is 1.225 $kg.m^{-3}$ at sea level, and $0.414 kg.m^{-3}$ at 10 km altitude.)

Density varies with temperature, with weather, and with altitude.

Intuitively, the dynamic pressure is the pressure that is needed to stop the movement of the air.

We have

$$L = q.S.C_L(\alpha)$$

$$D = q.S.C_D(\alpha)$$

Center of Pressure

The totality of aerodynamic forces on a wing can be summed into a single force working on a single acting point.

This acting point is called the center of pressure.

Its position is important for stability. A forward center of pressure will cause the plane to pitch up. A backward center of pressure will cause the plane to pitch down.

The center of pressure moves with angle of attack. It usually moves forward when angle of attack increases. Since this causes a pitch up moment, wings are usually unstable.

Pitching Moment

Instead of specifying center of pressure, one can also specify pitching moment.

Assume some fixed point p on the wing. Let x be the current position of the center of pressure.

The pitching moment M around p is defined as $M = (p - x) \times \overline{F}$, where $\overline{F}$ is the total aerodynamic force (lift and drag) on the wing.

Pitching moment is positive if it tries to obtain a pitch up (increase in angle of attack).

Pitching moment is usually modeled by the formula

$$M = q.c.S.C_M(\alpha),$$

where c is the chord length.

Aerodynamic Center

Theoretical models predict that if one takes $p = \frac{1}{4}c$, then $C_M(\alpha)$ does not depend on α.

In practice, this works only for a limited range of α, and the constant point often lies not exactly at $\frac{1}{4}$.

Anyway, the point p, for which $C_M(\alpha)$ is constant, or almost constant, or sometimes constant, is called the aerodynamic center of the wing.

(For NACA 0009, the aerodynamic center behaves pretty well between $\alpha = -10^\circ$ and $\alpha = 10^\circ$)

If $C_M(\alpha)$ is listed, it is usually based on $p = \frac{1}{4}$.

Sometimes, another point close to $\frac{1}{4}$ is chosen.

Table Data

The coefficient functions $C_L(\alpha)$ and $C_D(\alpha)$ have to be put in tables and interpolated.

For the pitching moment, one can either make a table of $C_X(\alpha)$ (relative position of center of pressure), or of $C_M(\alpha)$ (the pitching moment).

I think that $C_M(\alpha)$ is better, because $C_X(\alpha)$ is not well-defined when the lift is low. $C_X(\alpha)$ tends to take crazy values at low lift, which makes interpolation difficult.

Relating Moment Coefficient to Center of Pressure

Let P be the reference point, let $X(\alpha)$ be the position of center of pressure for α, let $M(\alpha)$ be the pitching moment for α.

$$(\ P - X(\alpha) \ )(L.\cos\alpha + D.\sin\alpha) = M(\alpha).$$

Let p be the position of P, expressed as fraction of chord length. Using the formulas on the previous slides, we get

$$(\ p - C_X(\alpha) \ ) \ c \ (\ q.S.C_L(\alpha).\cos\alpha + q.S.C_D(\alpha).\sin\alpha \ ) = q.c.S.C_M(\alpha).$$

Dividing by $q.S.c$ results in

$$(\ p - C_X(\alpha) \ )(\ C_L(\alpha) \cos\alpha + C_D(\alpha) \sin\alpha \ ) = C_M(\alpha).$$

This expression can be used to compute $C_M(\alpha)$. If one knows $C_M(\alpha)$, and needs to compute $C_X(\alpha)$, one can use

$$C_X(\alpha) = p - \frac{C_M(\alpha)}{C_L(\alpha).\cos\alpha + C_D(\alpha).\sin\alpha}$$

Performance: Gliding as Far as Possible

Suppose one is quietly flying, and the engines break. How does one get as far as possible?

Let ϑ be the gliding angle. Let W be the weight of the plane. We will try to minimize ϑ.

If we are gliding at angle ϑ, then $L = q.S.C_L(\alpha) = W.\cos\vartheta$, and $D = q.S.C_D(\alpha) = W.\sin\vartheta$. It follows that

$$\frac{q.S.C_D(\alpha)}{q.S.C_L(\alpha)} = \frac{C_D(\alpha)}{C_L(\alpha)} = \frac{W.\sin\vartheta}{W.\cos\vartheta} = \tan\vartheta.$$

The minimal value of $\tan\vartheta$ is obtained when $\frac{C_D(\alpha)}{C_L(\alpha)}$ is minimal.

Note that this calculation does not take the fuselage into account. In order to get correct numbers, one needs $C_L(\alpha)$, $C_D(\alpha)$ for the complete aircraft.

Performance: Flying as Efficiently as Possible

Since energy is force times distance (dotproduct), it follows that power is force times speed.

We assume that we are flying at fixed altitude, in equilibrium. It follows that

$$W = \frac{1}{2}\rho.V^2.S.C_L(\alpha).$$

Solving V results in

$$V = \sqrt{\frac{2W}{\rho.S.C_L(\alpha)}}$$

Since all force that contributes to the power used originates from drag, it follows that the required power equals $D.V =$

$$\frac{1}{2}\rho.V^2.S.C_D(\alpha).V = \frac{1}{2}\rho.S.C_D(\alpha).\sqrt{\frac{8.W^3}{\rho^3.S^3.C_L^3(\alpha)}}$$

This term can be simplified into

$$C_D(\alpha).\sqrt{\frac{2.W^3}{\rho.S.C_L^3(\alpha)}}$$

Since W, ρ, S are constant, we need to find the value of α for which

$$\frac{C_D(\alpha)}{\sqrt{C_L^3(\alpha)}}$$

is minimal.

One must be a bit careful interpreting this formula. It applies when one wants to fly as long as possible (distance does not matter), and it assumes that power consumption of the engine equals $D.V$. This does not apply to all (most) engines.

If one wants to fly a given distance, as effectively as possible, minimize D. The principles are sound.

Modeling the Complete Airplane

We want to model the complete airplaine (in two dimensions). Before we can do that, we need mechanics for rigid objects in two dimensions.

Mass Distribution

We assume that the rigid object is built-up from masses $m_1, \ldots, m_n$, which are at positions $\overline{x}_1, \ldots, \overline{x}_n$.

The total mass is defined as $M = \sum_{i=1}^{n} m_i$.

The mass center (also center of gravity) is defined as

$$\overline{C} = \frac{1}{M} \sum_{i=1}^{n} m_i \overline{x}_i.$$

The average speed is defined as

$$\overline{V} = \frac{1}{M} \sum_{i=1}^{n} m_i \overline{v}_i.$$

The moment of inertia/inertial moment relative to a point $\overline{p}$ is defined as

$$I_{\overline{p}} = \sum_{i=1}^{n} m_i \, \|\overline{x} - \overline{p}\|^2 = \sum_{i=1}^{n} m_i \, (\, (x_x - p_x)^2 + (x_y - p_y)^2 \,).$$

Mass Distribution (Continuous)

Assume a function $M(\overline{x})$ that represents mass density at position $\overline{x}$.

Total mass equals

$$M = \int_{\overline{x} \in \mathcal{R}^2} M(\overline{x}).d\overline{x}.$$

The mass center (also center of gravity) is defined as

$$\overline{C} = \frac{1}{M} \int_{\overline{x} \in \mathcal{R}^2} M(\overline{x}).\overline{x}.d\overline{x}.$$

The moment of inertia/inertial moment relative to a point $\overline{p}$ is defined as

$$I_{\overline{p}} = \int_{\overline{x} \in \mathcal{R}^2} M(\overline{x}).\|\overline{x} - \overline{p}\|^2.d\overline{x}.$$

Rigid Objects

If everything goes well, the airplane remains a rigid object. A rigid object is an object whose point masses preserve their relative positions.

Definition: A speed function is a function that maps space to speed. A speed function is rigid if it can be written in the form

$$\overline{V}(\overline{x}) = \overline{V}_0 + \overline{\omega} \times \overline{x}.$$

Speed functions can also be used in 3D-space. Since we are considering only 2D here, angular velocity will be represented by a single number ω, and $\overline{\omega} \times \overline{x}$ can be written as $(-\omega.x_2,\ \omega.x_1)$.

The speed $\overline{V}_0$ does not represent the speed of the object! It is the speed assigned to the object at position $(0, 0)$, but the object itself may be somewhere else.

Movement of Rigid Object under Force

We assume a sequence of forces $\overline{F}_1, \ldots, \overline{F}_k$, acting at positions $\overline{y}_1, \ldots, \overline{y}_k$, working on the rigid object.

The total force is defined as

$$\overline{F} = \sum_{i=1}^{k} \overline{F}_i.$$

Given a reference point $\overline{p}$, the total torque around $\overline{p}$ is defined as

$$\overline{T}_{\overline{p}} = \sum_{i=1}^{k} ((\overline{y}_i - \overline{p}) \times \overline{F}_i).$$

Because $\overline{F}_i, \overline{p}$ and $\overline{y}_i$ are two-dimensional vectors, the torque can be considered a single number.

Theorem: Assume a rigid object, whose masses $m_1, \ldots, m_n$ are at positions $\overline{x}_1, \ldots, \overline{x}_n$. Assume that $\overline{C} = 0$ for this rigid object.

Assume that the movement of the object is characterized by the rigid speed function $\overline{V}(\overline{x}) = \overline{V}_0 + \omega \times \overline{x}$.

Let $\overline{V}$ be the average speed of the rigid object. Then $\overline{V} = \overline{V}_0$.

proof:

$$\overline{V} = \frac{1}{M} \sum_{i=1}^{n} m_i \overline{v}_i = \frac{1}{M} \sum_{i=1}^{n} m_i (\overline{V}_0 + \omega \times \overline{x}_i) =$$

$$\frac{1}{M} \sum_{i=1}^{n} m_i \overline{V}_0 + \frac{1}{M} \sum_{i=1}^{n} m_i (\overline{\omega} \times \overline{x}_i) =$$

$$\overline{V}_0 + \omega \times \frac{1}{M} (\sum_{i=1}^{n} m_i \overline{x}_i) = \overline{V}_0.$$

(Choose a coordinate system with origin in $\overline{C}$)

Theorem: Assume a rigid object, whose masses $m_1, \ldots, m_n$ are at positions $\overline{x}_1, \ldots, \overline{x}_n$. Assume that $\overline{C} = 0$. Assume that movement of the object is characterized by the rigid speed function $\overline{V}(\overline{x}) = \overline{V} + \omega \times \overline{x}$. Assume that the total force on the object equals $\overline{F}$. Assume that the total torque around the mass center equals $T_{\overline{C}}$. Then

$$\begin{cases} T_{\overline{0}} = I_{\overline{0}}.\omega' \\ \overline{F} = M\,(\overline{V}' + \omega \times \overline{V}\ ). \end{cases}$$

The second equation is obtained from the familiar $\overline{F} = M\overline{V}'$. The term $\omega \times \overline{V}$ corrects acceleration for the fact that, when the object is moving and rotating at the same time, this causes a change in speed at the origin, which does not correspond to actual acceleration of the object.

$$\begin{cases} \omega' = \dfrac{T_{\overline{0}}}{I_{\overline{0}}} \\ \\ \overline{V}' = \dfrac{\overline{F}}{M} - \omega \times \overline{V} \end{cases} \tag{1}$$

If one writes out the components, one gets:

$$\begin{cases} \omega' = \dfrac{T_{\overline{0}}}{I_{\overline{0}}} \\ \overline{V}'_1 = \dfrac{\overline{F}_1}{M} + \omega.\overline{V}_2 \\ \overline{V}'_2 = \dfrac{\overline{F}_2}{M} - \omega.\overline{V}_1 \end{cases} \tag{2}$$

Either equation can be used to find ω' and $\overline{V}'$ if we know $T_{\overline{0}}$ and $\overline{F}$.

Before Equation 1 can be used, a coordinate system must be fixed. There are many ways to do this:

1. In Local Right Up (LRU) coordinates, the origin is fixed at a point on the ground. It is assumed that the earth is flat, infinite and two dimensional. The X-axis points horizontal to the right, and the Y-axis points up.

2. In body coordinates, the origin is at the mass center of the airplane. The X-axis points forward along the fuselage of the plane (the direction is usually well-defined). The Y-axis points up and is obtained by rotating the X-axis over $90°$ to the left.

Flight Model in LRU coordinates

At a given time point, the state of the airplane is determined by a position vector $\overline{x}$, a speed vector $\overline{v}$, a pitch angle ϑ, and an angular velocity ω.

In addition, the various components of the airplane may have state parameters, which may be important for stability analysis.

If the plane has an autopilot, it may have some internal state as well, which is important for stability analysis.

We need expressions for $\overline{F}$ and $T_{\overline{0}}$. Using those, we can get a differential equation for the complete airplane.

Aircraft model in LRU Coordinates

Gravity:

$$\overline{F}_G = (0, -M.g).$$

Gravity does not create torque around center of mass.

Gravity is a very well-behaved, easy to model force with only one disadvantage: It always points downwards.

Engines

It is assumed that the engines produce a force in forward direction:

$$\overline{F}_{thr} = (\cos\vartheta, \sin\vartheta) F_{thr}.$$

Determining F_{thr} is complicated. F_{thr} depends on the type of engine, on throttle selected by the pilot, on altitude and speed, and possibly on angle of attack. Engine data are usually not public.

In addition to force, engines produce a torque:

$$\overline{T}_{thr} = -F_{thr}.\overline{\mathrm{en}}_2$$

$\overline{\mathrm{en}}_2$ is the relative Y-position of the engines, which is negative if the engines are below the wing, and probably positive if the plane has high wings.

I assume that engine force is aligned with the body X-axis. If not, a correction term is needed.

Main Wings

The main wings are usually connected to the air frame under a positive angle ϑ_w.

The angle of attack of the wings equals $\alpha = \vartheta_w + \vartheta + \arctan(-\overline{v}_2, \overline{v}_1)$.

The lift- and drag- and moment coefficients equal $C_L(\alpha), C_D(\alpha), C_M(\alpha)$.

$$\overline{F}_w = \frac{1}{2}\rho\|\overline{v}\|^2 S(\ C_L(\alpha).\frac{(-\overline{v}_2, \overline{v}_1)}{\|\overline{v}\|} - C_D(\alpha).\frac{\overline{v}}{\|\overline{v}\|}\ ) =$$

$$\frac{1}{2}\rho S\|\overline{v}\|(\ C_L(\alpha)(-\overline{v}_2, \overline{v}_1) - C_D(\alpha)\overline{v}\ ).$$

In order to determine the torque resulting from the main wing, we need to know the point $\overline{w}$, on which it is fixed to the air frame. For simplicity, we take $\overline{w}$ as the point that was used in defining $C_M(\alpha)$. Then torque around $\overline{w}$ equals

$$M_w = \frac{1}{2}\,\rho\,S\,c\,\|\overline{v}\|^2 C_M(\alpha).$$

Torque around the mass center can be obtained from

$$T_w = M_w + \overline{w} \times \overline{F}_w.$$

Elevators

The elevators are little wings which are situated at relative position $(\overline{e}_1, \overline{e}_2)$. Almost certainly, $\overline{e}_1 < 0$, and $\overline{e}_2 > 0$.

The elevators are mounted to the air frame under an angle of attack α_e. This angle of attack usually can be adjusted by the pilot. This adjustment is called trimming. In addition, the elevators have flaps which change the profile. The flaps are connected to the controls of the pilot.

Because the elevators are at some distance from the mass center, the speed of the elevators $\overline{w}$ depends on $\overline{v}$ and ω as follows:

$$\overline{w} = \overline{v} + \omega \times \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} (\overline{e}_1, \overline{e}_2) =$$

$$\begin{cases} \overline{v}_1 - \omega(\overline{e}_1 \sin\vartheta + \overline{e}_2 \cos\vartheta) \\ \overline{v}_2 + \omega(\overline{e}_1 \cos\vartheta - \overline{e}_2 \sin\vartheta). \end{cases}$$

(If ϑ and $\overline{e}_2$ are small, we can use $\overline{w}_1 \approx \overline{v}_1$, and $\overline{w}_2 \approx \overline{v}_2 + \omega.e_1$.

The angle of attack of the elevators equals

$$\alpha = \vartheta_{\mathbf{e}} + \vartheta + \arctan(-\overline{w}_2, \overline{w}_1).$$

Once α and $\overline{w}$, are known, the same expression as for the main wings can be be used:

$$\overline{F}_e = \frac{1}{2}\rho S \|\overline{w}\| \left(C_L(\alpha)(-\overline{w}_2, \overline{w}_1) - C_D(\alpha)\overline{w} \right).$$

Torque around $\overline{e}$ (connection point of elevators) equals

$$M_e = \frac{1}{2}\,\rho\, S\, c\, \|\overline{w}\|^2 C_M(\alpha).$$

Torque around mass center can be obtained from

$$T_e = M_e + \overline{e} \times \overline{F}_e.$$

Fuselage

Modeling fuselage is difficult: The fuselage causes drag and lift.

Lift has a center of pressure. The pressure center is probably in front of the mass center. As a consequence, the fuselage contributes to pitch instability. (This effect also exists in the dimension that we are currently ignoring.)

Since the fuselage is much thicker than an airfoil, we need to know the pressure center for drag. It is safe to assume that it is in the middle at 0 angle of attack, but I have no idea, where it is at other angles of attack. Its position may be important for pitch stability.

For wings, one can find some data in books or on the internet. I found nothing useful yet for the fuselage.

Wheels

If the wheels touch the ground, they create force, which one may wish to model.

If they are not on the ground (and not retracted), the wheels may create resistance, which has to be modelled.

I think that this is easy:

Implementing Equation 1 with Euler Method

Let h be the step size. Compute

$$\begin{cases} \omega_{t+h} = \omega_t + h\dfrac{T_{\overline{0}}}{I_{\overline{0}}} \\ \overline{V}_{t+h} = \overline{V}_t + h(\dfrac{\overline{F}}{M} - \omega_t \times \overline{V}_t) \\ \vartheta_{t+h} = \vartheta_t + h.\omega_t \\ \overline{X}_{t+h} = \overline{X}_t + h.\overline{V}_t \end{cases}$$

It may seem that this is all, but it is not. The problem is that $\overline{V}$ has changed its meaning: At time t, $\overline{V}_t$ denoted the speed of the mass center of the object. At time $t+h$, $\overline{V}_{t+h}$ denotes the speed at the position where the mass center was at time t.

In order to correct $\overline{V}_{t+h}$ we have to apply the rigid speed function $\overline{V}_{t+h} + \omega_{t+h} \times \overline{x}$ at position $\overline{V}h$. The result is $\overline{V}'_{corr,t+h} = \overline{V}'_{t+h} + \omega \times \overline{V}h$. We see that $\overline{V}'_{corr,t+h} = \overline{V}_t + h\frac{\overline{F}}{M}$, so that the correction term disappears.

In general, one must be very very careful applying the correction term $\omega \times \overline{V}$. I have seen errors in text books, and also have made errors by myself.

If one uses different coordinate systems, e.g. body coordinates based on Euler angles, the situation gets even worse, because the correction term may be hard to recognize.