Exercises due on Friday 1st of February 11.15

January 29, 2002

1. **Minimal Propositional Logic**

   Give inhabitants of the following formulae:

   (a) $(A \to B \to C) \to (B \to A \to C)$. (Do not forget that $\to$ is rightassociative)

   (b) $(A \to B \to C) \to ((C \to \bot) \to A \to B \to \bot)$. (You don't need the definition of $\bot$ in this proof)

   (c) $(A \to B \to C) \to (A \to C) \to (B \to C)$.

   (d) $(A \to B \to C) \to (B \to A \to B \to A \to C)$.

2. **Inuitionistic Propositional Logic**

   Give inhabitants of the following formulae, assuming the declarations on the next page:

   (a) $A \lor (B \lor C) \to (A \lor B) \lor C$.

   (b) $A \land (B \land C) \to (A \land B) \land C$.

   (c) $\neg(A \lor B) \to (\neg A \land \neg B)$.

   (d) $(\neg A \lor \neg B) \to \neg(A \land B)$.

3. **Intuitionistic Predicate Logic**

   Give inhabitants of the following formulae:

   (a) $\Pi S{:}\,\mathrm{Set}\ \Pi s{:}\,S\ (\mathrm{equal}\ S\ s\ s)$.

   (b) $\Pi S{:}\,\mathrm{Set}\ \Pi s_1, s_2{:}\,S\ (\mathrm{equal}\ S\ s_1\ s_2) \to (\mathrm{equal}\ S\ s_2\ s_1)$.

   (c) $\Pi S{:}\,\mathrm{Set}\ \Pi s_1, s_2, s_3{:}\,S$
   $(\mathrm{equal}\ S\ s_1\ s_2) \to (\mathrm{equal}\ S\ s_2\ s_3) \to (\mathrm{equal}\ S\ s_1\ s_3)$.

   (d) Assume that the context contains

   $\mathrm{Nat}{:}\,\mathrm{Set}$
   $0{:}\,\mathrm{Nat}$
   $S{:}\,\mathrm{Nat} \to \mathrm{Nat}$
   $P{:}\,\mathrm{Nat} \to \mathrm{Prop}$

h: Πx: Nat
$$(P\ x) \to ((P\ (S\ x)) \to \bot) \to \bot.$$

In this context, construct a proof of
Πx: Nat
$$(P\ x) \to ((P\ (S\ (S\ x))) \to \bot) \to \bot.$$

Remark: The proof will start with λx: Nat $\cdots$ In this context, you will need two instances of h, namely $(h\ x)$ and $(h\ (Sx))$.

Intuitionistic First-Order Logic
$\top$: Prop trueintro: $\top$
$\vee$: Prop $\to$ Prop $\to$ Prop orintro1: ΠA, B: Prop A $\to$ (A $\vee$ B) orintro2: ΠA, B: Prop B $\to$ (A $\vee$ B) orelim: ΠA, B, C: Prop (A $\vee$ B) $\to$ (A $\to$ C) $\to$ (B $\to$ C) $\to$ C
$\wedge$: Prop $\to$ Prop $\to$ Prop andintro: ΠA, B: Prop A $\to$ B $\to$ (A $\wedge$ B) andelim1: ΠA, B: Prop (A $\wedge$ B) $\to$ A andelim2: ΠA, B: Prop (A $\wedge$ B) $\to$ B
$\bot := \Pi P$: Prop P
$\neg := \lambda A$: Prop $(A \to \bot)$
$\leftrightarrow := \lambda A, B$: Prop $(A \to B) \wedge (B \to A)$
$\exists$: ΠS: Set $(S \to$ Prop$) \to$ Prop existsintro: ΠS: Set ΠP: S $\to$ Prop Πs: S ((P s) $\to$ ($\exists$ S P)) existselim: ΠS: Set ΠP: S $\to$ Prop ΠC: Prop ($\exists$ S P) $\to$ (Πs: S (P s) $\to$ C) $\to$ C
equal: ΠS: Set $S \to S \to$ Prop eqrefl: ΠS: Set Πs: S (equal S s s) eqrepl: ΠS: Set $\Pi s_1, s_2$: Set ΠP: S $\to$ Prop (equal S s_1 s_2) $\to$ (P s_1) $\to$ (P s_2)