Natural Deduction for First-Order Logic

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There are several types of deduction systems. The two most important are:

- Natural Deduction: Natural Deduction tries to follow the natural style of reasoning. Most of the proof consists of forward reasoning, i.e. deriving conslusions, deriving new conclusions from these conclusions, etc. Occassionally hypotheses are introduced or dropped.
- Sequent Calculus: In sequent calculus, conclusions and premisses are treated in the same way. In the proof, the premisses are built up simultaneously with the conclusions. This is more unnatural for human readers, but it is technically simpler.

Natural Deduction

Rules for the constants:

$\top$-introduction:

$\ldots$

$\top$

$\bot$-elimination:

$\bot$

$\ldots$

A

Rules for $\neg$:

$\neg$-introduction:

$$\begin{array}{l|l} & \begin{array}{|l} A \\ \hline \ldots \\ \bot \end{array} \\ & \neg A \end{array}$$

$\neg$-elimination:

$$\begin{array}{|l} A \\ \ldots \\ \neg A \\ \ldots \\ \bot \end{array}$$

Rules for $\wedge$:

$\wedge$-elimination:

$A \wedge B$	$A \wedge B$
$\ldots$	$\ldots$
A	B

$\wedge$-introduction:

A

$\ldots$

B

$\ldots$

$A \wedge B$

Rules for $\vee$:

$\vee$-elimination: (reasoning by cases)

$$
\begin{array}{|l}
A \vee B \\
\ldots \\
\begin{array}{|l}
A \\ \hline
\ldots \\
C
\end{array} \\
\ldots \\
\begin{array}{|l}
B \\ \hline
\ldots \\
C
\end{array} \\
C
\end{array}
$$

Rules for $\vee$(continued):

$\vee$-introduction:

A	B
$\ldots$	$\ldots$
$A \vee B$	$A \vee B$

Rules for $\leftrightarrow$:

$\leftrightarrow$-elimination:

$A \leftrightarrow B$

$\ldots$

$A \rightarrow B$

$A \leftrightarrow B$

$\ldots$

$B \rightarrow A$

$\leftrightarrow$-introduction:

$A \rightarrow B$

$\ldots$

$B \rightarrow A$

$\ldots$

$A \leftrightarrow B$

Rules for $\rightarrow$:

$\rightarrow$-elimination:

$$
\begin{array}{|l}
A \\
\ldots \\
A \rightarrow B \\
\ldots \\
B
\end{array}
$$

$\rightarrow$-introduction:

$$
\begin{array}{|l}
\begin{array}{|l}
A \\
\hline
\ldots \\
B
\end{array} \\
A \rightarrow B
\end{array}
$$

Rules for $\forall$:

$\forall$-elimination:

$\forall x{:}X\ P(x)$

$\dots$

$P[x := t]$

$\forall$-introduction:

x is new object of type X

$\dots$

$P(x)$

$\forall x{:}X\ P(x)$

Rules for $\exists$:

$\exists$-elimination:

$\exists x{:}X\ P(x)$

$\ldots$

x is new object of type X

$P(x)$

$\ldots$

C

C

Rules for $\exists$(continued): $\exists$-introduction:

$P[x := t]$

$\ldots$

$\exists x{:}X\ P(x)$

Rules for $\approx$:

Equality reflexivity:

$\ldots$

$t \approx t$

Equality replacement:

$t_1 \approx t_2$

$\ldots$

$P(t_1)$

$\ldots$

$P(t_2)$

Remarks:

Small variations in the rules are possible. For example $\vee$-elim, $\exists$-elim, $\neg$-rules.

All operations take place on toplevel. Logic is not Algebra! Equality is not Equivalence!

The notation $A[x := t]$ means intelligent replacement.

What are the orders?

- Predicates that speak about domain objects are of 1-st order.
- Predicates that speak about objects of at most i-th order, are by themselves of $(i+1)$-th order.
- Functions that take and return domain objects are of 1-st order.
- Functions that take and return objects of at most i-th order, are by themselves of $(i+1)$-th order.

First-Order is not Enough for Most of Mathematics

- Induction Principle. For each predicate P :
  If $P(0)$ and $\forall x{:}\mathrm{Nat}\ P(x) \rightarrow P(x+1)$ are given, then
  $\forall x{:}\mathrm{Nat}\ P(x)$ may be concluded.
- Operations on functions, for example taking limits, differentiation, integration, Fourier-transform.
- Properties of functions, for example continuity, differentiability, surjectivity, etc.
- Properties of predicates, finiteness, enumerability, etc.

Some Fundamental Notions

First introduce the following notation

$$A_1, \ldots, A_p \vdash B,$$

for

$$B \text{ can be proven from } A_1, \ldots, A_p.$$

If it is not clear which proof system is meant, one can write things like

$$A_1, \ldots, A_p \vdash_{natded} B,$$

or

$$A_1, \ldots, A_p \vdash_{seqcalc} B.$$

Classical Logic vs. Intuitionistic Logic

There are two readings of logic, namely classical and intuitionistic.

Classical logic is truth-value based: One can imagine a possible state-of-affairs and then evaluate a logical formula in it. The result is either true or false.

States-of-affairs are usually called *interpretations*. An interpretation needs to specify three things:

1. A domain D, the set of objects about which one speaks.
2. For each n-ary function it needs to specify for each $(d_1, \ldots, d_n) \in D \times \cdots \times D$ the result.
3. For each n-ary predicate symbol, it needs to specify for each $(d_1, \ldots, d_n) \in D \times \cdots \times D$, the truth value.

If one states that formula F is true, one limits the possible interpretations to those where F is true.

A formula B is a *logical consequence* of formulae $A_1, \ldots, A_p$ if in every interpretation where all of $A_1, \ldots, A_p$ are true, B is also true. Notation for this is

$$A_1, \ldots, A_p \models B.$$

If one accepts interpretations, then a proof system can (and should) have the following properties:

Soundness:

If

$$A_1, \ldots, A_p \vdash B$$

then

$$A_1, \ldots, A_p \models B.$$

Completeness:

If

$$A_1, \ldots, A_p \models B,$$

then

$$A_1, \ldots, A_p \vdash B.$$

Natural Deduction, as we presented, has the soundness property, but not the completeness property.

Examples of formulue that cannot be proven are

$$A \vee \neg A, \ (\neg\neg A \rightarrow A),$$

(without premisses)

or

$$P(0), \neg P(s(s(s(0)))) \vdash \exists x{:}\,\mathrm{Nat}\ (\ P(x) \wedge \neg P(s(x))\ ).$$

Principle of Excluded Middle

$\neg\neg A$

$\ldots$

A

or

$\ldots$

$A \vee \neg A$

Intuitionistic Logic

If one leaves out the law of excluded middle, the examples remain unprovable. In that case one obtains intuitionistic logic, which is not based on interpretations, but on algorithms.

The idea is that one should speak only about things that one can compute.

Interpretations are problematic, because one cannot compute truth-values in infinite interpretations.