Sequent Calculi for First-Order Logic

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The most important types of deduction systems are:

- Natural Deduction: Natural Deduction tries to follow the natural style of reasoning. Most of the proof consists of forward reasoning, i.e. deriving conslusions, deriving new conclusions from these conclusions, etc. Occassionally hypotheses are introduced or dropped.
- Sequent Calculus: In sequent calculus, conclusions and premisses are treated in the same way. In the proof, the premisses are built up simultaneously with the conclusions. This is more unnatural for human readers, but it is technically simpler.

Sequent Calculi are obtained by explicit reasoning about the $\vdash$-relation.

Sequent calculi are more general than natural deduction. There is only one natural deduction calculus, (ignoring small differences), but there are many sequent calculi for first-order logic. Most other proof systems can be straightforwardly translated into a sequent calculus.

This is also possible with natural deduction.

Translation of Natural Deduction

A sequent is an object of the form

$$\Gamma \vdash A.$$

Here Γ consists of formulae, and variable declarations. Formula A is the conclusion.

Translation of the $\vee$-rules

$\vee$-introduction: From $\Gamma \vdash A$ derive $\Gamma \vdash A \vee B$.

$\vee$-introduction: From $\Gamma \vdash B$ derive $\Gamma \vdash A \vee B$.

$\vee$-elimination: From

$$\Gamma \vdash A \vee B,$$

$$\Gamma, A \vdash C,$$

$$\Gamma, B \vdash C$$

derive

$$\Gamma \vdash C.$$

Translation of the $\exists$-rules

$\exists$-introduction: From

$$\Gamma \vdash P(t)$$

derive

$$\Gamma \vdash \exists x{:}X \;\; P(x).$$

$\exists$-elimination:

$$\Gamma \vdash \exists x{:}X \;\; P(x),$$

and

$$\Gamma, x{:}X, p(x) \vdash C$$

derive

$$\Gamma \vdash C.$$

Translation of the $\forall$-rules

$\forall$-introduction: From

$$\Gamma, x{:}\, X \vdash P(x)$$

derive

$$\Gamma \vdash \forall x{:}\, X\ \ P(x).$$

$\forall$-elimination: From

$$\Gamma, \forall x{:}\, X\ \ P(x)$$

derive

$$\Gamma \vdash P(t).$$

Structural Rules

Cut: From

$$\Gamma \vdash A,$$

and

$$\Gamma, A \vdash B$$

derive

$$\Gamma \vdash B.$$

Exercise: Translate the other rules.

Sequent Calculus for Classical Logic

A multiset is a set that can distinguish how often an element occurs in it, (or alternatively it is a list that cannot see the order of its elements).

Examples:

$$A \vee B,\ A \wedge B,\ A \wedge B$$

$$A \vee B,\ A \wedge B,\ C \to D.$$

$$A \wedge B,\ A \vee B,\ A \wedge B.$$

The first and the last multiset are equal.

A Sequent is a construction of the form

$$\Gamma \vdash \Delta.$$

Here both Γ and Δ are multisets of formulae.

The meaning is: Whenever all of the Γ are true, at least one of the Δ is true.

Axioms:

$$(\text{axiom}\ )\ \frac{}{\Gamma, A \vdash \Delta, A}$$

Structural Rules:

$$(\text{weakening left})\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad (\text{weakening right})\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

$$(\text{contraction left})\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad (\text{contraction right})\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Rules for the constants:

$$(\top\text{-left }) \ \frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta} \qquad (\top\text{-right }) \ \frac{}{\Gamma \vdash \Delta, \top}$$

$$(\bot\text{-left }) \ \frac{}{\Gamma, \bot \vdash \Delta} \qquad (\bot\text{-right }) \ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}$$

Rules for $\neg$:

$$(\neg\text{-left }) \ \frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta} \qquad (\neg\text{-right }) \ \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

Rules for $\wedge$ and $\vee$:

$$(\wedge\text{-left}\ )\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\wedge\text{-right}\ )\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$(\vee\text{-left}\ )\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \qquad (\vee\text{-right}\ )\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

(one can see from this, that premisses and conclusions are treated in completely the same way)

Rules for $\rightarrow$ and $\leftrightarrow$:

$$(\rightarrow\text{-left}) \ \frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad\qquad (\rightarrow\text{-right}) \ \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

$$(\leftrightarrow\text{-left}) \ \frac{\Gamma, A \rightarrow B,\ B \rightarrow A \vdash \Delta}{\Gamma, A \leftrightarrow B \vdash \Delta}$$

$$(\leftrightarrow\text{-right}) \ \frac{\Gamma \vdash \Delta, A \rightarrow B \qquad \Gamma \vdash \Delta, B \rightarrow A}{\Gamma \vdash \Delta, A \leftrightarrow B}$$

Rules for the quantifiers:

$$(\forall\text{-left }) \ \frac{\Gamma, P[x := t] \vdash \Delta}{\Gamma, \forall x{:}\, X \ \ P(x) \vdash \Delta} \qquad\qquad (\forall\text{-right }) \ \frac{\Gamma \vdash \Delta, P[x := y]}{\Gamma \vdash \Delta, \forall x{:}\, X \ \ P(x)}$$

$$(\exists\text{-left }) \ \frac{\Gamma, P[x := y] \vdash \Delta}{\Gamma, \exists x{:}\, X \ \ P(x) \vdash \Delta} \qquad\qquad (\exists\text{-right }) \ \frac{\Gamma \vdash \Delta, P[x := t]}{\Gamma \vdash \Delta, \exists x{:}\, X \ \ P(x)}$$

The t is an arbitrary term of type X. The y is a variable of type X, which is not free in Γ, Δ.