1 Exercises for the Lecture Automated Theorem Proving

(To be completed by June 21st, 18.00)

1. Consider the Knuth-Bendix order based on

$$f \succ g \succ a \succ b \succ c,$$

and all symbols have equal weight.

Construct the set of clause that can be obtained by superposition, equality resolution and equality factoring from the following set of ground clauses:

$$\begin{aligned}
&[f(a) \approx f(b),\quad f(a) \not\approx c],\\
&[f(f(a)) \approx g(a),\quad f(a) \not\approx a],\\
&[f(a) \not\approx f(c),\quad f(f(a)) \not\approx f(f(b))],\\
&[g(f(a)) \approx f(a),\quad g(f(a)) \approx f(b)],\\
&[g(f(a)) \approx f(b),\quad f(a) \not\approx f(a)],\\
&[g(f(a)) \not\approx g(f(a)),\quad f(a) \approx f(b)].
\end{aligned}$$

2. Consider the KBO from the previous question.

   (a) Does $[a \approx b]$ make $[f(a) \approx f(b)]$ redundant?

   (b) Do $[a \approx b]$ and $[b \approx c]$ make $[a \approx c]$ redundant? And $[f(a) \approx f(b)]$?

   (c) Do $[a \approx b],\ [c \approx d]$ make $[f(a, c) \approx f(b, d)]$ redundant?

   (d) Does $[f(a) \not\approx f(b)]$ make $[a \not\approx b]$ redundant?

   (e) Does $[f(b) \not\approx f(c)]$ make $[b \not\approx c,\quad f(a) \approx c]$ redundant?

3. The KBO, as we defined it in the lecture, is not the original definition by Knuth and Bendix.

The problem with the KBO from the lecture is that it cannot direct associativity axioms with variables. (and such axioms occur very often)

Which clauses are derivable by superposition from

$$[(X + Y) + Z \approx X + (Y + Z)]$$

and

$$[p(X + (Y + Z), A + (B + C))]?$$

Is any of these clauses redundant?

4. Let $\succ$ be the following order: (The real Knuth-Bendix order)

   (a) If $\#t_1 > \#t_2$, then $t_1 \succ t_2$.

   (b) If $\#t_1 = \#t_2$, then write t_1 in the form $t_1 = f(\alpha_1, \ldots, \alpha_n)$, and write t_2 in the form $t_2 = g(\beta_1, \ldots, \beta_m)$.

    i. If $f \succ g$, then $t_1 \succ t_2$.
    ii. If $f = g$, then let i be the smallest index for which $\alpha_i \not\approx \beta_i$. If $\alpha_i \succ \beta_i$, then $t_1 \succ t_2$.

5. Show that $\succ$ from the previous question is a reduction order.

6. Saturate the clauses of question 3 using the new Knuth-Bendix order. Delete redundant clauses. What do you see?

7. Addition on the natural numbers can be axiomatized by

$$\mathbf{Ax1}: \ [\neg N(X),\ \neg N(Y),\ X + Y \approx Y + X],$$

$$\mathbf{Ax2}: \ [\neg N(X),\ X + 0 \approx X].$$

In order to prove that + is commutative, one has show (by induction) that

$$\mathbf{L1}: \ \forall x : N \ \ 0 + X \approx X,$$

$$\mathbf{L2}: \ \forall x, y : N \ \ s(X) + Y \approx s(X + Y).$$

Using those lemmas, one can show that

$$\mathbf{L3BASE}: \ \forall x : N \ X + 0 \approx 0 + X,$$

$$\mathbf{L3STEP}: \ \forall x, y : N \ X + Y \approx Y + X \rightarrow X + s(Y) \approx s(Y) + X.$$

(The first is the base case of the inductive proof, the second is the step case)

Show that Ax1, Ax2, L1, L2 imply L3BASE, and that Ax1, Ax2, L1, L2 imply L3STEP using superposition.

(One has to negate the goal, replace the variables by constants c, d, and show that the result is unsatisfiable)

Use the KBO (the simple one) based on

$$c \succ d \succ + \succ s \succ 0 \succ N.$$