Using the Primitive Rules of HOL

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The goal of these exercises is to convince you that it is possible to obtain the standard logical rules in the primitive calculus of HOL. The primitive rules can be found in the slides, or on pages 52-53 of 'HOL manual, version 1.1'. The definitions of the operators are on page 54.

Normally, the user will not deal directly with the primitive rules of HOL, because they are quite unpleasant, and the tactics of HOL are very good. But it is still interesting to puzzle out the deriviations of the standard rules.

1. Remember that in HOL, definitions are just equalities. Because of this, we first need to establish some equality reasoning, before it makes sense to define logical operators.

   (a) Prove that if t_2 is obtained from t_1 by a one-step β-reduction, then $\vdash t_1 = t_2$ is provable. (We did this one in class)

   (b) Prove the weakening rule: If $\Gamma \vdash A$ is provable, and $\Gamma \subseteq \Delta$, then $\Delta \vdash A$ is provable. (I am actually not sure if this provable)

   (c) Prove the cut rule: If $\Gamma \vdash A$ and $\Delta, A \vdash B$ are provable, then $\Gamma, \Delta \vdash B$ is provable.

   (d) Prove that if the sequent $\Gamma \vdash t_1 = t_2$. is provable, then $\Gamma \vdash t_2 = t_1$ is provable.

2. Next, we prove some rule involving the truth constant:

   (a) Prove $\Gamma \vdash \top$.

   (b) Prove $\Gamma \vdash (A = \top)$ from $\Gamma \vdash A$.

   (c) Prove $\Gamma \vdash A$ from $\Gamma \vdash (A = \top)$.

   Remember that HOL internally does not distinguish $=$ from $\leftrightarrow$.

3. Some standard logical rules involving $\wedge$:

   (a) Prove $\Gamma \vdash A \wedge B$, using $\Gamma \vdash A$ and $\Gamma \vdash B$.

   (b) Prove $\Gamma \vdash A$ from $\Gamma \vdash A \wedge B$.

   (c) Prove $\Gamma \vdash B$ from $\Gamma \vdash A \wedge B$.

4. (a) Prove $\Gamma \vdash A \to B$ using $\Gamma, A \vdash B$.
   (b) Prove $\Gamma, A \vdash B$ using $\Gamma \vdash A \to B$.

5. (a) Prove $\Gamma \vdash A \vee B$ from $\Gamma \vdash A$.
   (b) Prove $\Gamma \vdash A \vee B$ from $\Gamma \vdash B$.
   (c) Prove $\Gamma, A \vee B \vdash C$ from $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$.

6. (a) Prove $\Gamma \vdash P(t)$ from $\Gamma \vdash \forall x\ P(x)$ for arbitrary term t.
   (b) Prove $\Gamma \vdash \forall x\ P(x)$ from $\Gamma \vdash P(x)$, assuming that x is not free in Γ.
   (c) Prove $\Gamma \vdash \exists x\ P(x)$ from $\Gamma \vdash P(t)$.
   (d) Prove $\Gamma,\ \exists x P(x) \vdash C$ from $\Gamma, P(x) \vdash C$, assuming that x does not occur in C or Γ.