http://www.ii.uni.wroc.pl/~sle/open/numan/na-spline.pdf

# Numerical analysis

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## V. Cubic spline interpolation

### 1. Natural cubic interpolating spline function

**Definition 1.1** **Given** $n$, $a = x_0 < x_1 < \ldots < x_n = b$ and a function $f$, the ***natural cubic interpolating spline function*** (**NCISF**, in short) is a function $s$, which has the following properties:

(1) $s$, $s'$ and $s''$ are continuous in $[a, b]$,

(2) for $k = 1, 2, \ldots, n$, the function $s$ is the interval $[x_{k-1}, x_k]$ identical with a certain polynomial $\pi_k$ of degree at most 3:

$$s(x) = \pi_k(x) \qquad (x \in [x_{k-1}, x_k]),$$

(3) the function $s$ interpolates the function $f$ at the nodes $x_k$, i.e.,

$$s(x_k) = f(x_k) \qquad (k = 0, 1, \ldots, n),$$

(4) the function $s''$ vanishes at the ends of the interval $[a, b]$: $s''(a) = s''(b) = 0$.

THEOREM 1.2. *For arbitrary $n \in \mathbf{N}$, $a = x_0 < x_1 < \ldots < x_n = b$ and the function $f$ there exists a unique natural cubic interpolating spline function $s$. In any interval $[x_{k-1}, x_k]$ ($k = 1, 2, \ldots, n$) the spline function is given by*

$$\begin{aligned}(1.1)\qquad s(x) =& h_k^{-1}\left[\frac{1}{6}M_{k-1}(x_k - x)^3 + \frac{1}{6}M_k(x - x_{k-1})^3\right.\\ &\left.+\left(f(x_{k-1}) - \frac{1}{6}M_{k-1}h_k^2\right)(x_k - x) + \left(f(x_k) - \frac{1}{6}M_k h_k^2\right)(x - x_{k-1})\right]\end{aligned}$$

***The moments***

$$(1.2)\qquad M_k := s''(x_k) \qquad (k = 0, 1, \ldots, n;\ M_0 = M_n = 0)$$

*satisfy the linear system*

$$(1.3)\qquad \lambda_k M_{k-1} + 2M_k + (1 - \lambda_k)M_{k+1} = d_k \quad (k = 1, 2, ..., n-1),$$

*where*

$$\lambda_k := h_k/(h_k + h_{k+1}), \quad h_k := x_k - x_{k-1}, \quad d_k := 6f[x_{k-1}, x_k, x_{k+1}].$$

It can shown that the system (1.3) has a unique solution $M_1, \ldots, M_{n-1}$, which can computed by using the following algorithm.

ALGORITHM 1.3. *Compute recursively the auxilliary quantities $p_1, p_2, \ldots, p_{n-1}$, $q_0, q_1, \ldots, q_{n-1}$, and $u_0, u_1, \ldots, u_{n-1}$ by*

$$(1.4)\qquad q_0 := u_0 := 0,$$

$$(1.5)\qquad \left.\begin{aligned} p_k &:= \lambda_k q_{k-1} + 2,\\ q_k &:= (\lambda_k - 1)/p_k,\\ u_k &:= (d_k - \lambda_k u_{k-1})/p_k \end{aligned}\right\} \quad (k = 1, 2, \ldots, n-1).$$

*Then*

$$M_{n-1} = u_{n-1}, \tag{1.6}$$

$$M_k = u_k + q_k M_{k+1} \qquad (k = n-2, n-3, \ldots, 1). \tag{1.7}$$

As for the error estimation in approximating the function $f$ by NCISF, we have the following result.

Theorem 1.4. *Assume that $f$, $f'$, $f''$, $f'''$, $f^{(4)}$ are continuous in the interval $[a, b]$. Let $s$ be NCISF for the function $f$, and nodes $x_0, x_1, \ldots, x_n$ ($a = x_0 < x_1 < \ldots < x_n = b$). Then*

$$\max_{a \le x \le b} |f^{(r)}(x) - s^{(r)}(x)| \le C_r\, h^{4-r} \max_{a \le x \le b} |f^{(4)}(x)| \qquad (r = 0, 1, 2),$$

*where*

$$C_0 := 5/384, \quad C_1 := 1/24, \quad C_2 := 3/8,$$
$$h := \max_i h_i, \quad h_i := x_i - x_{-1} \qquad (i = 1, 2, \ldots, n).$$

**Example 1.5** For the Runge's function $f(x) = 1/(25x^2 + 1)$ $(-1 \le x \le 1)$, and the equidistant nodes, we obtain the following results.

Table V.1. Error in approximation of the Runge's function by NCISF

| $n$ | 10 | 20 | 40 | 80 | 160 |
|---|---|---|---|---|---|
| $h$ | 0.2 | 0.1 | 0.05 | 0.025 | 0.0125 |
| $\max_{-1 \le x \le 1} \lvert f(x) - s(x) \rvert$ | 0.022 | 0.0032 | $2.77 \cdot 10^{-4}$ | $1.60 \cdot 10^{-5}$ | $9.63 \cdot 10^{-7}$ |

□

## 2. An extremal property of the NCISF

For a fixed function $f$, define

$$y_k := f(x_k) \qquad (k = 0.1 \ldots, n).$$

NCISF has the following extremal property.

Theorem 2.1 (Holladay). *Let $\mathfrak{F}$ be a class of the functions $F$ which have continuous second derivative in the interval $[a, b]$, and satisfy*

$$F(x_k) = y_k \qquad (k = 0, 1, \ldots, n). \tag{2.1}$$

*The integral*

$$\int_a^b [F''(x)]^2 \, dx \tag{2.2}$$

*reaches the minimal value for $F$ being NCISF. Moreover,*

$$\int_a^b [s''(x)]^2 \, dx = \sum_{k=1}^{n-1} (f[x_k, x_{k+1}] - f[x_{k-1}, x_k]) \, M_k. \tag{2.3}$$

We give geometric interpretation of the above property of NCSIF. The *curvature* of a curve defined by a function $y = g(x)$ is useful for describing the geometric properties of the curve. The curvature at the point $x$ is given by

$$\kappa(x) := \frac{g''(x)}{(1 + [g'(x)]^2)^{3/2}}.$$

*Total curvature* in the interval $[a, b]$ is defined by

$$K := \int_a^b [\kappa(x)]^2 \, dx.$$

Now, if we assume that $|g'(x)| \ll 1$ for $x \in [a, b]$, then the second derivative of the function $g(x)$ is a good approximation of the curvature at the point $x$. Hence, the total curvature in the interval $[a, b]$ can be aproximated by

$$\int_a^b [g''(x)]^2 \, dx \approx \int_a^b [\kappa(x)]^2 \, dx.$$

The Holladay's theorem implies a practical importance of natural spline functions. Theorem says that the natural spline minimizes the total curvature, i.e., has the smoothest graph, in the class $\mathfrak{F}$ of the functions interpolating the given function $f$.

## 3. Examples

**Example 3.1** Consider the function $f(x) = \arctan x$ with $-4 \le x \le 4$, and take 9 equidistant nodes

$$x_k = k - 4 \quad (k = 0, 1, \ldots, 8).$$

Figures 1 and 2 show the results of approximating the function $f(x)$ using the Lagrange interpolating polynomial of degree 8, and the NCSIF. Notice that in both cases the same data is used.

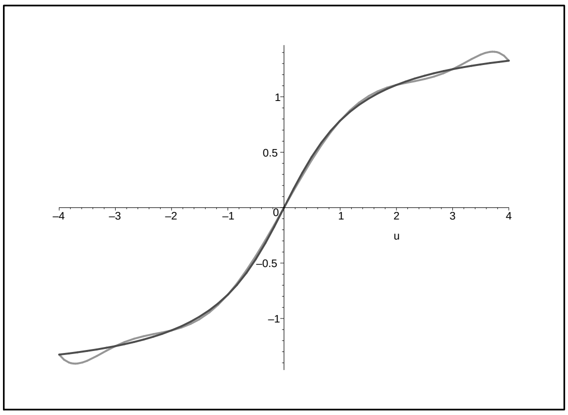


Figure 1. Graphs of $\arctan x$ and the interpolation polynomial $L_8(x)$

□

**Example 3.2** Now, let us consider the Runge's function (cf. Figure 3)

$$f(x) = \frac{1}{1 + 25x^2} \quad (-1 \le x \le 1).$$

Also this time we take equidistant nodes

$$x_k = -1 + \frac{2k}{n} \quad (k = 0, 1, \ldots, n).$$

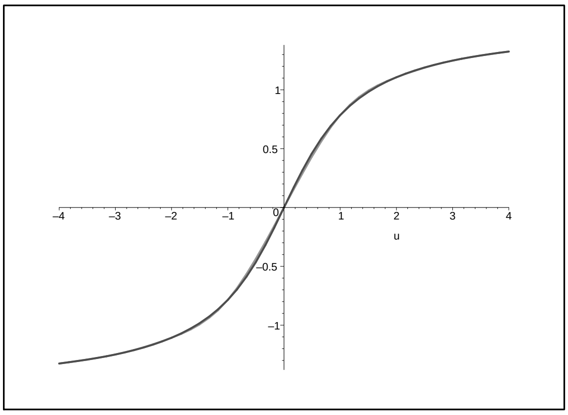


FIGURE 2. Graphs of $\arctan x$ and the natural spline $s(x)$

Figures 4 and 6 show the results of approximating the function $f(x)$ using the Lagrange interpolating polynomial of degree $n = 10$, and the NCSIF using the same data, i.e., the values of the function $f(x)$ at 11 equidistant nodes.

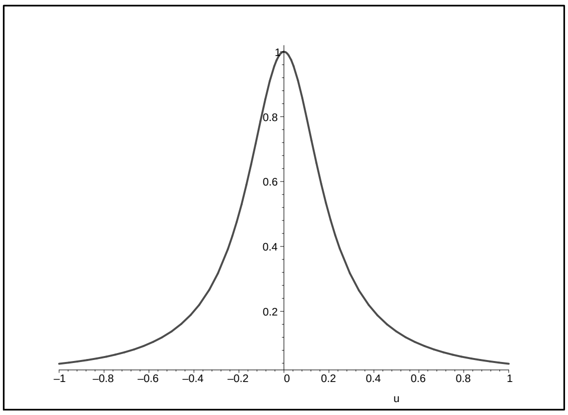


FIGURE 3. Graph of Runge's function

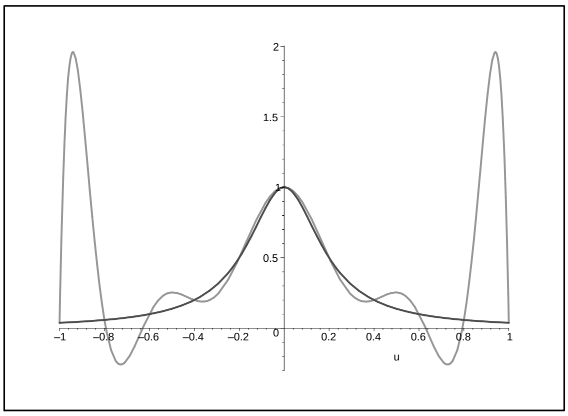


FIGURE 4. Graphs of Runge's function and the interpolating polynomial $L_{10}$

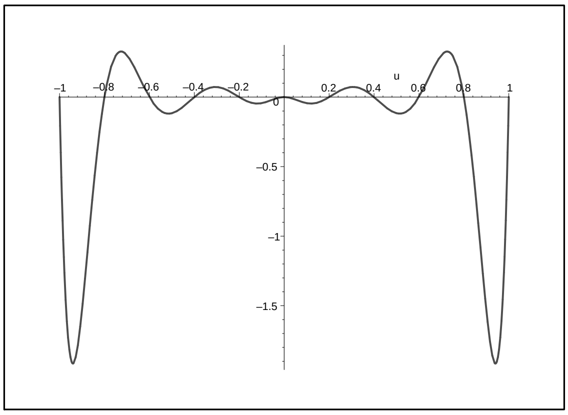


Figure 5. Error function $f - L_{10}$

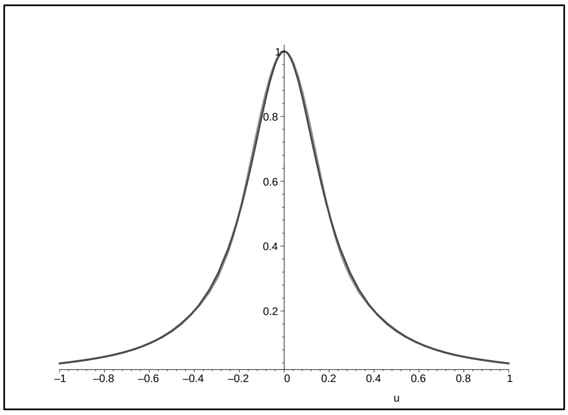


Figure 6. Graphs of Runge's function and the spline function $s(x)$

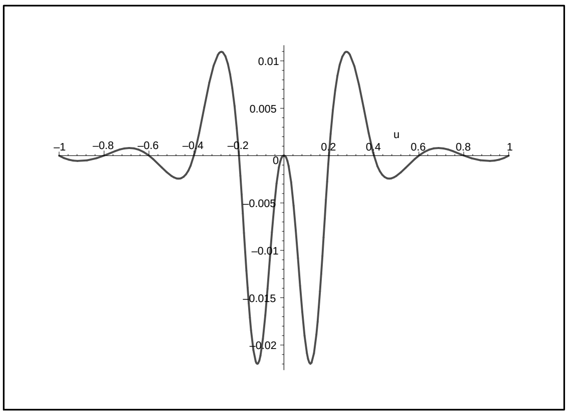


Figure 7. Error function $f - NCSIF$

$\square$